DOCUMENT RESUME

ED 173 136 SE 027 969

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TITLE Mathematics for Junior High School. Commentary for Teachers. Volume II (Part 2). Preliminary Edition.

INSTITUTION Stanford Univ., Calif. School Mathematics Study Group.

SPONS AGENCY National Science Foundation, Washington, D.C.

PUB DATE 60

NOTE 124p.; For related documents, see SE 027 968-970 and ED 130 876; Pages 205-236 missing from document prior to being shipped to EDRS for filming; Contains occasional light and broken type; Best copy available

EDRS PRICE MF01/PC05 Plus Postage.

DESCRIPTORS Congruence; Curriculum; *Curriculum Guides; *Geometry; *Instruction; Junior High Schools; Mathematics Education; *Number Concepts; *Probability; Secondary Education; *Secondary School Mathematics; Symmetry

IDENTIFIERS *School Mathematics Study Group

ABSTRACT

This is part two of a three-part manual for teachers using SMSG junior high school text materials. Each chapter contains an introduction and a collection of sample test questions. Each section contains a discussion related to the topic at hand and answers to all the exercises. Chapter topics include: (1) symmetry, congruence, and the Pythagorean Property; (2) real numbers; (3) permutations and selections; (4) probability; and (5) similar triangles and variation. However, some pages from the third chapter and most of the fourth chapter are missing. (MP)

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MATHEMATICS FOR JUNIOR HIGH SCHOOL

Commentary for Teachers

Volume II (Part 2)

(preliminary edition)

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Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

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NOTE TO TEACHERS

Based on the teaching experience of over 100 junior high school teachers in all parts of the country and the estimates of authors of the revisions (including junior high school teachers), it is recommended that teaching time for Part 2 be as follows:

Chapter	Approximate number of days
5	10
6	15
7	13
8	10
9	20
Total	68

Teachers are urged to try not to exceed these approximate time allotments so that pupils will not miss the chapters at the end of the course. Some classes will be able to finish certain chapters in less than the estimated time.

Throughout the text, problems, topics, and sections which were designed for the better students are indicated by an asterisk (*). Items starred in this manner should be used or omitted as a means of adjusting the approximate time schedule.

Chapter 5

SYMMETRY, CONGRUENCE, AND THE PYTHAGOREAN PROPERTY

This chapter continues with some geometric concepts. Symmetry with respect to a line is introduced. The only objective is to acquaint students with the word and its meaning. The geometry of the tenth grade will build upon this simple concept.

Congruence makes further use of the constructions taught in the last chapter. The approach is intuitive for the most part although there are one or two opportunities for the student to try an informal proof. The word"proof"is avoided in the student text, however.

The Pythagorean Property is presented along the lines found in many eighth grade texts but those students who are able are provided with an informal proof of the property.

It is estimated that most classes should cover this material in nine or ten days.

5-1. Symmetry

This Section introduces the concept of symmetry of a plane figure with respect to a line in order to prepare the pupil for ideas about congruent triangles. Symmetry with respect to a point and symmetry with respect to a plane are relegated to the starred exercises because they are not used in the subsequent development.

The pupil is to obtain his basic ideas of symmetry from class discussion of figures obtained by cutting from folded sheets of paper.

Answers to Class Exercises 5-1

1. (a) Triangular
   (c) Yes
   (d) 1, 3, 0

2. (b) Parallelogram

   (c) Yes. The halves fit. Yes. 2.

3. Yes. The lines bisecting the angles at the center are also axes of symmetry. 6.

4. Yes. Every diameter is an axis of symmetry. The number of axes of symmetry is greater than any number you might name. (If students are aware of the word "infinite," it is correct to say that there are an infinite number of axes of symmetry.)

5. Yes. The line perpendicular to $\overline{AB}$ at its midpoint is an axis of symmetry.

   2.

   $\overleftrightarrow{AB}$ is called the major axis because $\overline{AB}$ is larger than the segment cut on the minor axis. The minor axis is the line perpendicular to $\overline{AB}$ at its midpoint.

Answers to Exercises 5-1

1. 2. For a square, 4.
2. 3.
3. 4.
4. (a) 1, (b) 1,
   (c) 2, (d) 2,
   (e) 4, (f) 2,
   (h) 0, (i) 3.
5. Yes. Yes.
6. The two folds.

*7. A plane figure is symmetrical with respect to a point if for every line through the point the only intersections with the figure are pairs of points such that the two members of a pair are equally distant from the point of symmetry.

c, d, e, f.

*8. Sphere, cone, prisms, pyramid, many fruits when cut in half such as apples, bananas and grapefruit.

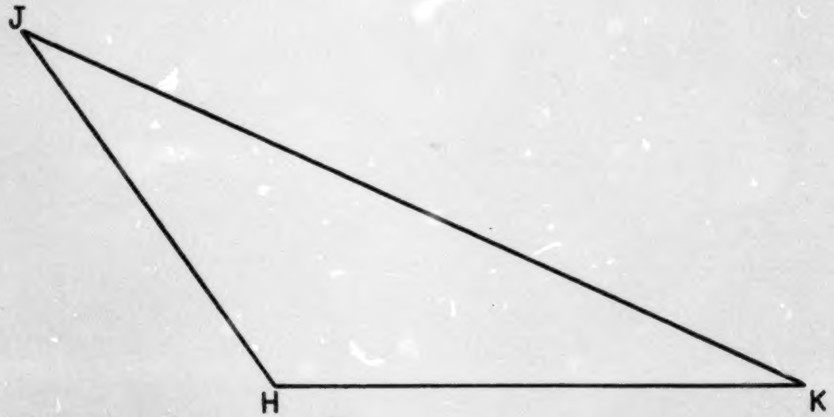
5-2. Congruent Triangles

Background for this chapter is built in the previous chapter. Congruent triangles are drawn with ruler, protractor, and compass but are not named as such. Notice that the word "congruent" is used to describe segments and angles formerly described as equal.

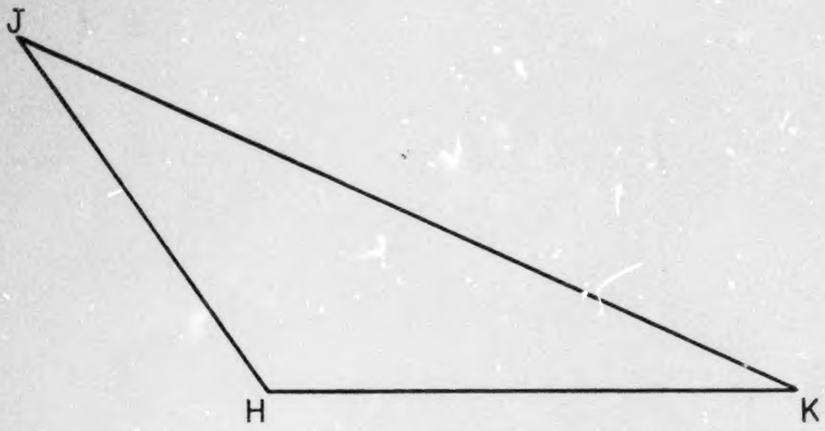
In this section correspondence is really a matter of choice. You are given, say, two triangles ABC and RST . A correspondence between the vertices can be set up in any one of six ways. Once such a correspondence has been set up, we decide that corresponding sides shall be opposite corresponding angles. With this understanding, a correspondence of vertices determines a correspondence of sides. Another way to set up a correspondence would be to start with the sides, and have it induce a correspondence of vertices. Usually we choose to set up a correspondence so that congruent angles correspond or congruent sides correspond.

Answers to Exercises 5-2a

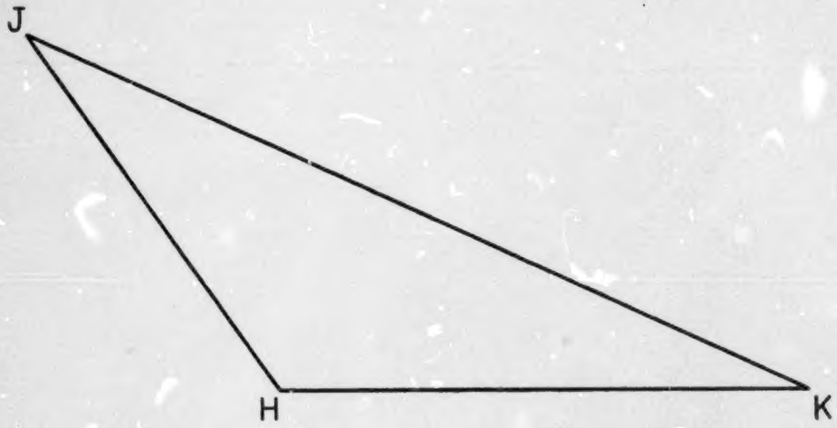
1. (a)



(b)

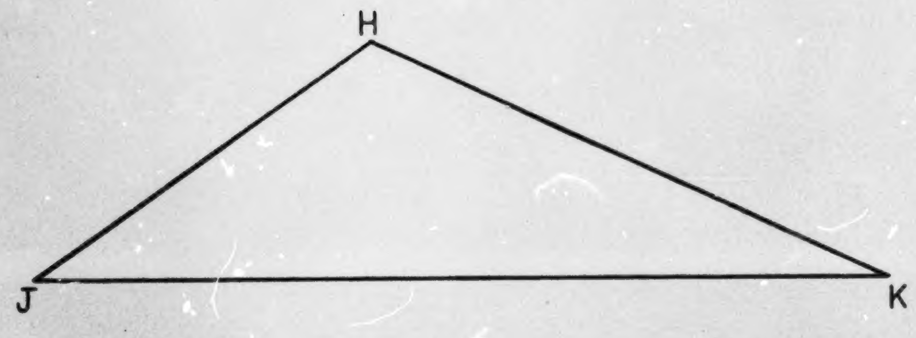


(c)

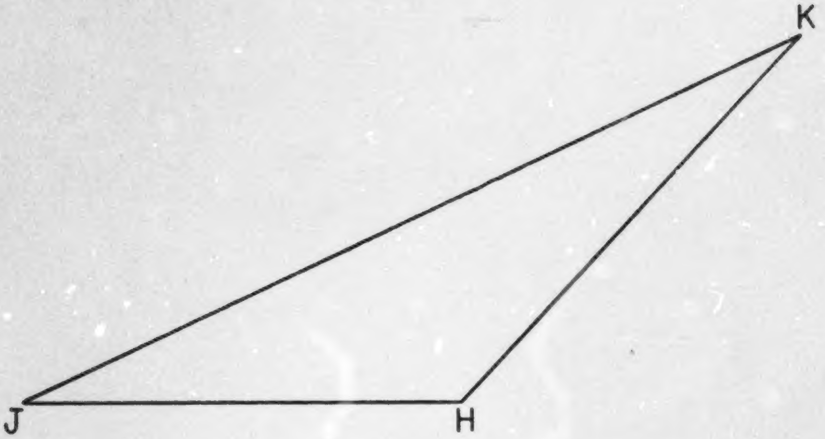


(d) Yes. Property S.S.S., Property A.S.A., and Property S.A.S.

2.



3.



Answers to Exercises 5-2b

1. (a) $\overline{AB} \cong \overline{BC}$, $\overline{AD} \cong \overline{CD}$. Be sure that the pupils understand the word "respectively."
   (b) $\overline{BD} \cong \overline{BD}$. Common side.
   (c) Yes. Property S.S.S.
   (d) Yes. If two triangles are congruent, then each pair of corresponding angles is congruent.
2. (a) $\overline{EF} \cong \overline{HJ}$, $\overline{FG} \cong \overline{JK}$, $\overline{EG} \cong \overline{HK}$.
   (b) Yes. Property S.S.S.
   (c) Yes. Property: If two triangles are congruent, then each pair of corresponding angles is congruent.
3. (a) Yes. $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$.
   (b) No. There is not a pair of sides equal.
   (c) Triangles would be congruent. Property A.S.A.

4. (a) Extends line XZ to make $\overline{ZR} \cong \overline{XZ}$. Yes.

(b) Yes. Property S.A.S.

(c) Same length as $\overline{QR}$ which can be measured.

5. (a) $\overline{QS} \cong \overline{TV}$, $\overline{QR} \cong \overline{TU}$, $\overline{SR} \cong \overline{VU}$

(b) $\overline{XY} \cong \overline{CB}$, $\overline{XZ} \cong \overline{AB}$, $\overline{ZY} \cong \overline{CA}$

(c) $\overline{XZ} \cong \overline{ZR}$, $\overline{ZY} \cong \overline{ZQ}$, $\overline{XY} \cong \overline{QR}$

6. (a) $\angle A \cong \angle E$, $\angle B \cong \angle F$, $\angle C \cong \angle D$.

(b) $\angle 1 \cong \angle 2$, $\angle X \cong \angle R$, $\angle Y \cong \angle Q$.

7. (a) Yes.

(b) Property S.A.S. or Property A.S.A.

8. (a) $\angle 1 \cong \angle 2$ because they are corresponding angles.

(b) Yes, they are vertical angles.

(c) Since $\angle 1 \cong \angle 2$ and $\angle 2 \cong \angle 3$, $\angle 1 \cong \angle 3$.

9. (a) Yes, corresponding angles.

(b) Vertical angles. Yes, they are congruent.

(c) They are congruent.

(d) $\angle 6 \cong \angle 7$ because they are vertical angles.
$\angle 4 \cong \angle 5$ in a manner similar to $\angle 1$ and $\angle 3$.

(e) No, one side is also needed.

(f) Yes, $\overline{AB} \cong \overline{CD}$ because they are opposite sides of a parallelogram.

(g) $\angle 1 \cong \angle 3$

$\angle 4 \cong \angle 5$

$\overline{AB} \cong \overline{CD}$

Therefore $\triangle ABE \cong \triangle CDE$ because of property A.S.A.

Hence $\overline{BE} \cong \overline{DE}$ and $\overline{AE} \cong \overline{CE}$ and the diagonals bisect each other.

*10. Triangle ECF $\cong$ Triangle EDF by Property S.S.S.

$\angle CEO \cong \angle OED$ because if two triangles are congruent, then each pair of corresponding angles is congruent.

$\overline{EO} \cong \overline{EO}$

Triangle COE $\cong$ triangle OED by Property S.A.S.

$\overline{CO} \cong \overline{OD}$ because if two triangles are congruent, then each pair of corresponding sides is congruent.

$\angle 1 \cong \angle 2$ because if two triangles are congruent, then each pair of corresponding angles is congruent.

*11. (a) Triangle ECF $\cong$ triangle EDF
Triangle COE $\cong$ triangle DOE
Triangle COF $\cong$ Triangle DOF

(b) $\overline{CE} \cong \overline{DE}$
$\overline{CF} \cong \overline{FD}$
$\overline{EF} \cong \overline{EF}$
$\overline{CO} \cong \overline{OD}$
$\overline{EO} \cong \overline{EO}$
$\overline{OF} \cong \overline{OF}$

(c) $\angle CEO \cong \angle DEO$
$\angle ECO \cong \angle EDO$
$\angle 1 \cong \angle 2$
$\angle CFO \cong \angle DFO$
$\angle FCO \cong \angle FDO$
$\angle COF \cong \angle DOF$
$\angle ECF = \angle EDF$

5-3. The Right Triangle

This section develops the concept of the Pythagorean Property intuitively. Those classes that are capable of a more rigorous proof of this property of the right triangle will find a development given in the next section. All that is essential for future work in this course is given in this section.

Students have been introduced to the classification of triangles by sides. This is reviewed at this time and the classification by angles is given as a basis for the work with right triangles.

Two special right triangles are used for the intuitive development. The 3-4-5 triangle is used both for understanding and also to give a historical background. An isosceles right triangle strengthens the understanding of the Pythagorean relationship. After the student has made observations using these two triangles, the Pythagorean Property is stated with a note that the student has seen it in only two special cases but that it is true for all right triangles. For many eighth graders this is sufficient.

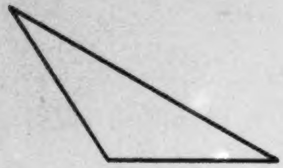
The necessity for square roots requires the introduction of both the concept and the symbol at this time. Only the positive square root and symbol are used but no point is made of the fact. Some students may be curious about the use of the word "positive;" the teacher could discuss the negative root but suggest that at this time they will use only the positive root and should leave a real study of negative roots for later work.

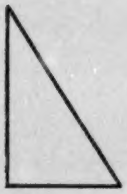
The only method presented for finding non-rational roots is the table. The writers recognize that this limits the problems that can be solved, but it seems more logical to develop a method for extracting roots in the next chapter, Real Numbers.

It should be noted by the teacher that the next section, 5-4, is starred but the Exercises 5-4 are not. All of the problems but number 7 can be used as a supplementary set of exercises even though the class did not work through the development given in the section.

Answers to Exercises 5-3a

1. Examples are given; there are many correct triangles.





2. (a) $5^2 = 4^2 + 3^2$

$25 = 16 + 9$

$25 = 25$

(b) $13^2 = 5^2 + 12^2$

$169 = 25 + 144$

$169 = 169$

(c) $25^2 = 7^2 + 24^2$

$625 = 49 + 576$

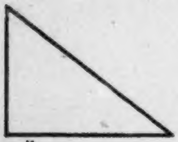
$625 = 625$

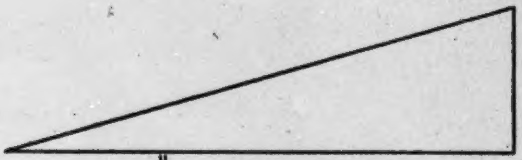
(d) $20^2 = 16^2 + 12^2$

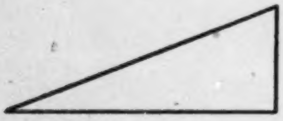
$400 = 256 + 144$

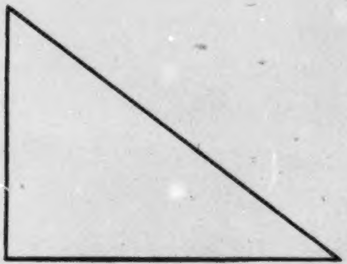
$400 = 400$

3.

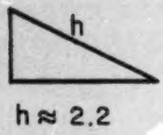

$\frac{1}{4}$" = 1 Unit

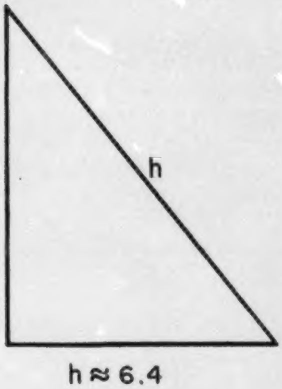

$\frac{1}{8}$" = 1 Unit

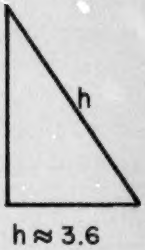

$\frac{1}{8}$" = 1 Unit


$\frac{1}{8}$" = 1 Unit

4.







5. (a) Area is 5 sq. cm.

(b) Area is 41 sq. cm.

(c) Area is 13 sq. cm.

Answers to Exercises 5-3b

1. (a) $\sqrt{5} \approx 2.236$

(b) $\sqrt{41} \approx 6.403$

(c) $\sqrt{13} \approx 3.606$

2. (a) App. 2.236" (d) App. 7.810 yd.

(b) App. 6.403' (e) App. 9.487'

(c) App. 3.606" (f) App. 2.162 units

3. (a) 12'

(b) 10'

(c) 36'

4. (a) 17 ft.

(b) 44 ft.

5. $16\frac{1}{2}$ ft.

6. 14 ft.
7. App. 6.4 ft.
8. 58 ft.
9. App. 85 ft.
10. The number of units in each case is $\sqrt{2}$ which is approximately 1.4.
11. The number of units is $\sqrt{3}$ which is approximately 1.7.

---

*5-4 One Proof of the Pythagorean Property

The word "proof" is used in the title of this section but avoided in the rest of the section. Students are asked to "show" that something is true. While the development leads to a logical proof, it is given in an informal manner more suitable for students of this age. Students who follow this work will probably need considerable help from the teacher; a satisfying sense of achievement will reward those students capable of understanding this approach.

The proof that is presented is, of course, not the one attributed to the Pythagoreans, since that proof is much too difficult at this time. The dissection of congruent squares in two different ways is the basis for the proof given. It would be advisable for the teacher to be thoroughly familiar with this development before teaching this to a class or allowing a few students to work on it alone or in a small group.

Answers to Class Exercises 5-4

1. $(a + b)^2 = a^2 + 2ab + b^2$

$$(4 + 3)^2 = 4^2 + 2(4)(3) + 3^2$$

$$7^2 = 16 + 24 + 9$$

$$49 = 49$$

2. $(2 + 6)^2 = 2^2 + 2(2)(6) + 6^2$

$8^2 = 4 + 24 + 36$

$64 = 64$

Answers to Questions in Proof

1. Two acute angles in a right triangle.
2. Corresponding angles of congruent triangles.
4. Sum of the measures of angles on one side of a line.
5. $m(\angle 2) + m(\angle 4) = 90$
6. Addition Property for equations.

Answers to Exercises 5-4

1. (a) 8.888

   (b) 9.592

   (c) 26

   (d) 75

2. 15 inches

3. (a) In $\triangle$ ATD, $\overline{AD}$ is congruent to $\overline{DB}$ in $\triangle$ BCD (Construction)

   In $\triangle$ ATD, $\angle$TAD is congruent to $\angle$DBC in $\triangle$ BCD (Right Angles)

   In $\triangle$ ATD, $\angle$ADT is congruent to $\angle$BDC in $\triangle$ BCD (Vertical $\angle$s)

   $\therefore \triangle$ ATD $\cong \triangle$ BCD by A.S.A. Property.

   (b) $\overline{BC}$

   (c) 42 ft.

4. 68 ft.

5. 4.472 ft.

6. 26 ft.

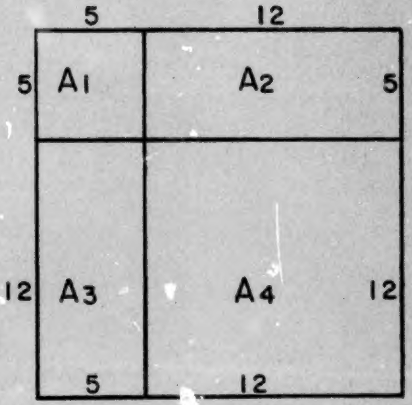
7. For Figure 1,

$$A = A_1 + A_2 + A_3 + A_4$$

$$= 5^2 + 5(12) + 5(12) + 12^2$$

$$= 25 + 60 + 60 = 144$$

$$= 289$$



5
12
5
A1
A2
5
12
A3
A4
12
5
12

Figure 1

For Figure 2,

$$A = T_1 + T_2 + T_3 + T_4 + A_{PQRS}$$

Since all Δs are congruent by S.A.S.,

$$T_1 + T_2 + T_3 + T_4 = 4(T_1)$$

$$A = 4(T_1) + A_{PQRS}$$

$$= 4(\frac{1}{2}\cdot 5\cdot 12) + A_{PQRS}$$

$$= 120 + A_{PQRS}$$

Since Fig. 1 $\cong$ Fig. 2

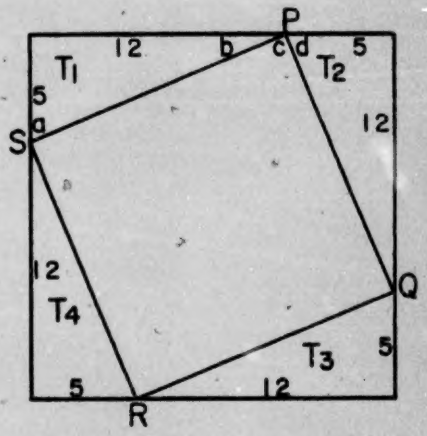
$$289 = 120 + A_{PQRS}$$

$$169 = A_{PQRS} \quad \text{(Equality Prop.)}$$

but in Fig. 1, $A_1 + A_4 = 25 + 144$

$$A_1 + A_4 = 169$$

So $A_1 + A_4 = A_{PQRS}$ and A_1 equals the measure of the square on the short side and A_4 equals the measure of the square on the long side of T_1.



P
12
b
c
d
5
T1
T2
5
a
S
12
12
Q
T4
T3
5
5
12
R

Figure 2

In PQRS, $\overline{PQ} \cong \overline{QR} \cong \overline{RS} \cong \overline{SP}$ because they are corresponding sides of congruent triangles.

$m(\angle a) + m(\angle b) = 90$ (They are the acute angles in a right triangle.)

$\angle a \cong \angle d$ (corresponding angles in congruent triangles)

$\therefore m(\angle b) + m(\angle d) = 90$

But $m(\angle d) + m(\angle c) + m(\angle d) = 180$

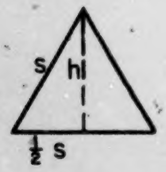
$\therefore m(\angle c) = 180 - 90$

and $m(\angle c) = 90$

In a similar way, the other angles in the quadrilateral can be shown to have measures of 90, and PQRS is a square.

A_{PQRS} has been proved equal to $A_1 + A_4$.

8. BRAINBUSTER:



$$h^2 = s^2 - (\tfrac{1}{2}s)^2$$

$$h^2 = s^2 - \tfrac{1}{4}s^2$$

$$h^2 = \tfrac{3}{4}s^2$$

$$h = \sqrt{\tfrac{3}{4}s^2}$$

(At this stage of development, this form should be acceptable.)

$$h = \frac{s}{2}\sqrt{3}$$

---

5 - 5 Quadrilaterals

This section is used as a summary and review section. No new ideas are presented. The concepts of this chapter are utilized in various relationships found in quadrilaterals. As a reminder, the various quadrilaterals are named and defined. Problem 9 in the exercises is quite lengthy. The teacher may wish to omit some of the figures given, or allow more than one day for the exercises.

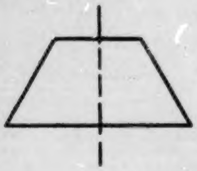
Answers to Exercises 5-5

1. (c) Rectangle and (d) Square

2. (a) 2

   (b) 4

3. Yes



4. Yes, any rectangle or square.

5. (a) 360°. All quadrilaterals can be separated into two triangles by a diagonal. The sum of the measures of the angles in a triangle is 180, so, for two triangles, it must be 360.

   (b) Parallelograms and the subsets, rectangles and squares.

6. 80 inches

7. 25 feet

8. (a) a and c   (b) a and e

   b and d   b and f

   e and g   d and h

   f and h   c and g

   (c) a and g   The reasons are given for $\angle a$ and $\angle g$.

   b and h $\angle a \cong \angle c$ (vertical angles)

   c and e $\angle c \cong \angle g$ (corresponding angles)

   d and f $\therefore \angle a \cong \angle g$

   A similar statement could be made for each of the other pairs in this set.

9. **Figure A**

(a) 8

(b) No.

Figure B

(a) 8

(b) Yes. △ ABC , △ CDA
△ ABD , △ BCD
△ AOD , △ COB
△ AOB , △ COD

Figure C

(a) 8

(b) Yes. △ ABC , △ ADC
△ ABD , △ BCD
△ AOB , △ BOC , △ COD , △ DOA

Figure D

(a) 8

(b) No.

Figure E

(a) 8

(b) Yes. △ ABC , △ CDA
△ ABD , △ BCD
△ AOB , △ COD
△ AOD , △ COB

Figure F

(a) 8

(b) Yes. Same △s as in Fig. C

(c) Proof of congruence can be given for two sets of congruent triangles. One or the other applies in all cases of congruence.

1. Prove $\triangle ABC \cong \triangle CDA$

   $\overline{AB} \cong \overline{CD}$ (opposite sides of a parallelogram)

   $\overline{BC} \cong \overline{DA}$ (opposite sides of a parallelogram)

   $\overline{AC} \cong \overline{AC}$ Same segment

   $\therefore \triangle ABC \cong \triangle CDA$ (because of Property S.S.S.)

2. Prove $\triangle ABO \cong \triangle DCO$

   $\overline{AB} \cong \overline{CD}$ (opposite sides of a parallelogram)

   $\angle ABO \cong \angle CDO$; (alternate interior angles formed by parallel lines and a transversal. See part (c) Problem 8)

   $\angle BAO \cong \angle DCO$

   $\therefore \triangle ABO \cong \triangle DCO$ because of Property A.S.A.

   If students remember that the diagonals of a parallelogram bisect each other, these triangles can be proved congruent by use of Property S.S.S. also.

Sample Questions

Select from this list. There are too many questions for one test.

True--False

(F) 1. All congruent figures are symmetrical.

(T) 2. The two halves of figures that are symmetrical with respect to a line are congruent.

(F) 3. A trapezoid always has an axis of symmetry.

(F) 4. The capital letter C has two axes of symmetry.

(T) 5. Every circle is symmetrical with respect to all its diameters and to its center.

(F) 6. Two triangles are congruent if three angles of one are respectively congruent to the three angles of the other.

(T) 7. Two rectangles are congruent if the bases are congruent and the altitudes are congruent.

(T) 8. Two triangles are congruent if two sides and the included angle of one are congruent to two sides and the included angle of the other.

(F) 9. Two triangles are congruent if two angles and a side of one are congruent to two angles and a side of the other.

(F) 10. Right triangles have two right angles.

(F) 11. The hypotenuse of a right triangle is on one ray that forms the right angle.

(T) 12. The length of the hypotenuse is less than the sum of the lengths of the other two sides of a right triangle.

(T) 13. An isosceles triangle may also be a right triangle.

(F) 14. An equilateral triangle may also be a right triangle.

(T) 15. A triangle whose sides measure 80 , 84 and 116 is a right triangle.

Completion and Multiple Choice

(e) 16. A trapezoid must have

- (a) Two pairs of parallel sides.
- (b) At least one pair of congruent sides.
- (c) Diagonals that always bisect each other.
- (d) One right angle.
- (e) None of these.

(c) 17. Rectangles are included in:

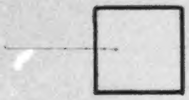
- (a) The set of squares.
- (b) The set of trapezoids.
- (c) The set of parallelograms.
- (d) The set of pentagons
- (e) None of these.

(d) 18. The diagonals of some figures bisect each other. In which of these groups is this true for all figures of the group?

- (a) Quadrilaterals, parallelograms and squares.
- (b) Trapezoids, parallelograms and rectangles.
- (c) Quadrilaterals, parallelograms and rectangles.
- (d) Parallelograms, rectangles and squares.
- (e) None of these.

(a) 19. All of these figures have at least one axis of symmetry.

1.



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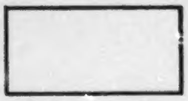

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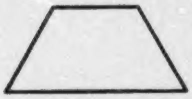

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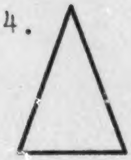

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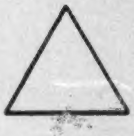


Select the group in which all figures have more than one axis of symmetry.

(a) 1, 2, and 5.

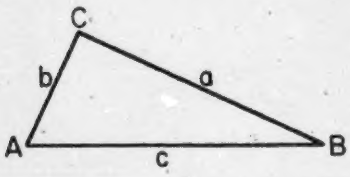
(b) 2, 3, and 4.

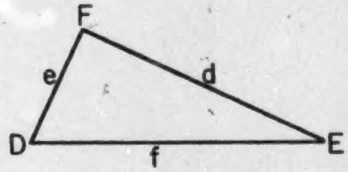
(c) 2, 3, and 5.

(d) 1, 2, and 3.

(e) All of these.

(c) 20. Two triangles, ABC and DEF are





congruent if the parts named are congruent.

(a) $\angle A$ and $\angle D$, $\angle B$ and $\angle E$, side a and side d .

(b) $\angle C$ and $\angle F$, side b and side e , side c and side f .

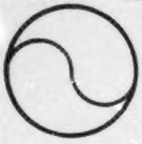
(c) $\angle B$ and $\angle E$, side c and side f , side a and side d .

(d) $\angle B$ and $\angle E$, $\angle C$ and $\angle F$, side c and side f .

(e) None of these.

(6) 21. A regular hexagon has ______ lines of symmetry.

(point)
(line) 22. This figure represents symmetry with respect to a ____________ but not to a ________.



(corresponding parts) 23. If two figures are congruent, then ____________ ____________ are congruent.

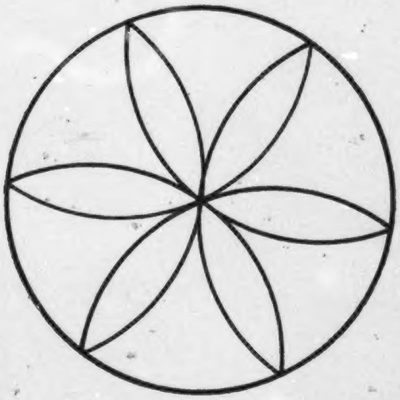
24. The Pythagorean Property states that the area of ____ ____

(ANSWER-- The square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.)

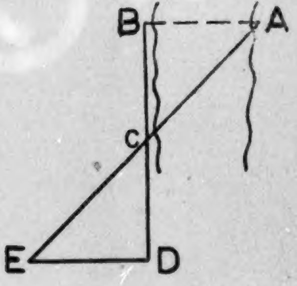
(7) 25. The hypotenuse of a right triangle is 25 units in length, and one side is 24 units in length. The other side is ___________ units in length.

Problems. Answers for the problems are at the end.

26. Draw all axes of symmetry in this figure.



27. In order to find the width of a creek, Bob located points A and B directly opposite each other. From point B he marked $\overline{BD}$ at right angles to $\overline{AB}$. Point C is the midpoint. Bob then marked $\overline{DE}$ so that $\overline{DE}$ is perpendicular to $\overline{BD}$ and so that A, C and E all lie on $\overleftrightarrow{AE}$.

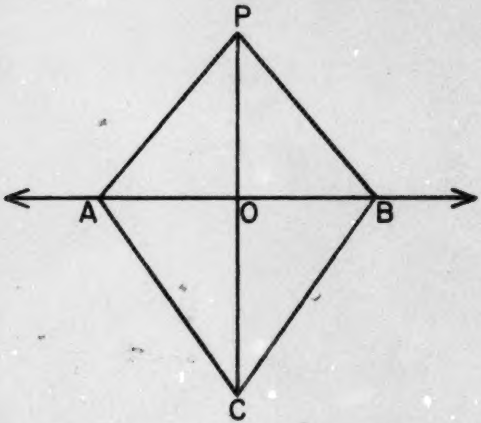


(a) If the length of $\overline{CD}$ is 15 ft., of $\overline{DE}$ is 36 ft., and of $\overline{CD}$ is 39 ft., how wide is the creek?

(b) This method of measure is based upon the congruence of $\triangle ABC$ and $\triangle EDC$. Show that these triangles are congruent.

28. The length of the hypotenuse of a right triangle is 41 inches. One side has a length of 9 inches. Find the third side.

29. In the figure a perpendicular is constructed from point P to $\overleftrightarrow{AB}$. Segments $\overline{AP}$ and $\overline{PB}$ are drawn. $\overline{AO} \cong \overline{OB}$.



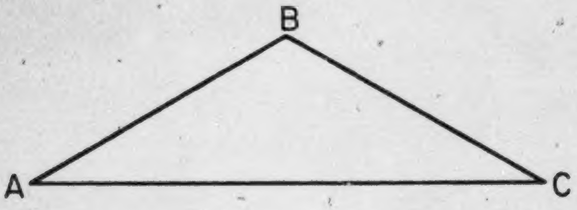
(a) Is $\triangle APO$ congruent to $\triangle OPB$? Why?

(b) Segments $\overline{AC}$, $\overline{OC}$, $\overline{BC}$, $\overline{OA}$ and $\overline{OB}$ form some triangles. Use a number sentence to tell which triangles formed by these segments are congruent.

(c) Name the triangle which is congruent to $\triangle CBP$.

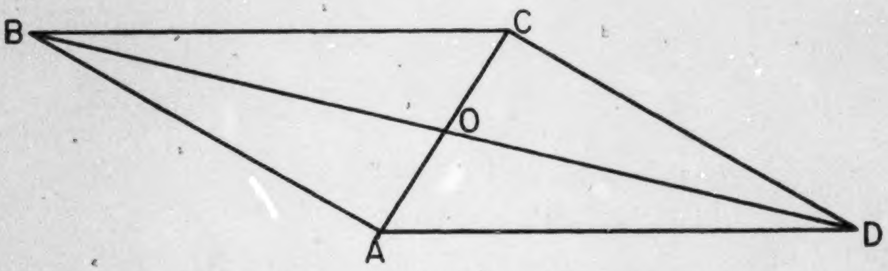
(d) Use number sentences to show the congruence of each pair of corresponding sides in triangles CAO and CBO .

30. Construct a triangle EFG congruent to triangle ABC



Mark the pairs of congruent angles and pairs of congruent sides.

31. In the parallelogram ABCD :



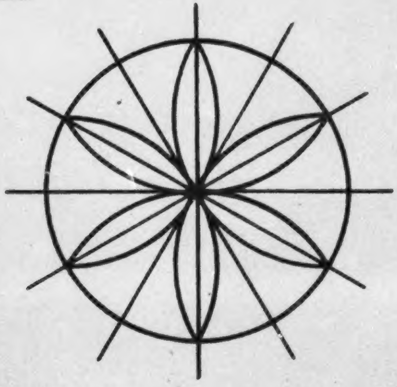
(a) List as many pairs of congruent triangles as you can.

(b) Use mathematical symbols to show the congruence of each pair of corresponding angles in triangles AOB and DOC ?

(c) Is $AD \cong BC$? Why?

Answers to Test Problems

26.



27. (a) 36 feet

(b) $\overline{BC} \cong \overline{CD}$ (construction)

$\angle ACB \cong \angle ECD$ (vertical angles)

$\angle ABC \cong \angle EDC$ (all right angles are congruent)

$\therefore \triangle ABC \cong \triangle EDC$, property A.S.A.

28. 40 inches.

29. (a) Yes. 3 sides are congruent respectively.

(b) $\triangle AOC \cong \triangle BOC$

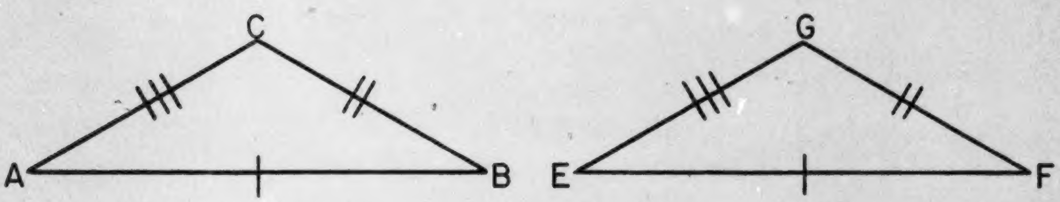
(c) $\triangle CAP$

(d) $\overline{OC} \cong \overline{OC}$

$\overline{OA} \cong \overline{OB}$

$\overline{AC} \cong \overline{BC}$

30.



31. (a) $\triangle ADC \cong \triangle CBA$

$\triangle ADB \cong \triangle CBD$

$\triangle ADO \cong \triangle CBO$

$\triangle ABO \cong \triangle CDO$

(b) $\angle BAO \cong \angle DCO$

$\angle ABO \cong \angle CDO$

$\angle AOB \cong \angle COD$

(c) Yes, opposite sides of a parallelogram or corresponding sides of congruent triangles.

Chapter 6

REAL NUMBERS

This Chapter is intended to give a first brief and intuitive approach to the real number system. The time needed for the average class is estimated to be 15 days.

The spirit of the chapter is in keeping with the approach outlined in the appendices of the Report of the Commission on Mathematics of the College Entrance Examination Board, Chapter 1, Algebra, Section 3.

Principal emphases have been placed on three main points:

First, the irrational numbers do in fact exist, and are as "real" as the rational numbers with which the pupils now have some familiarity. The first introduction of an irrational number by geometrical construction of a point on the number line should strengthen this conviction, as should the approach via decimals.

Second, great stress is placed upon the one-to-one correspondence between the points of a line and the real numbers.

Third, the important property of density is treated in detail. It leads naturally to a development of the possibility of approximating an irrational number by rationals.

The development of decimal representations for rationals is intended to strengthen the pupil's grasp of the rational number system as well as to provide a plausible basis for the introduction of irrationals.

As a by-product of the treatment the pupil should be expected to have gained a fair mastery of the properties of the rational number system, and of the real number system. He should be expected to state the properties in his own words. In particular the importance of density and closure should be continually emphasized.

The treatments of the real number system in the following references will provide much worthwhile supplementary material.

1. Bell, E.T. MEN OF MATHEMATICS: New York. Simon and Shuster, 1937.
   Life of Georg Cantor, Chap. 29, pp. 555-579.
2. National Council of Teachers of Mathematics, Twenty-Third Yearbook: INSIGHTS INTO MODERN MATHEMATICS. Washington, D.C., 1957. Chapter II, The Concept of Number.
3. National Council of Teachers of Mathematics, Twenty-Fourth Yearbook: THE GROWTH OF MATHEMATICAL IDEAS, Grades K-12; Washington, D.C., 1959. Chapter 2, Number and Operation; Chapter 11, Promoting the Continuous Growth of Mathematical Concepts.
4. Courant, Richard and Robbins, Herbert. WHAT IS MATHEMATICS? New York. Oxford University Press, 1953. Chapters 1 and 2.
5. Richardson, M. FUNDAMENTALS OF MATHEMATICS: New York. Macmillan, 1941.
6. College Entrance Examination Board; Report to the Commission on Mathematics, Appendices: College Entrance Examination Board, c/o Educational Testing Service, Box 592, Princeton, N.J. Part 1, Algebra. See especially pp. 28-35, a classroom approach to Irrational Numbers.
7. Gamow, George. ONE, TWO, THREE ... INFINITY. New York. Viking Press, 1947. Chapter 1, Big Numbers.
   See especially pp. 14-23.

Answers to Questions in Section 6-1

(a) ___ , ___ , N - 1 , N + 1 .

(b) Yes. One, because N - 1 would be zero and zero is not a counting number.

(c) Yes. No. The smallest is 1 .

(d) (1) Yes (2) No (3) Yes (4) No.

(e) 2 , 999 , none, No.

Zero is neither positive or negative.

Answers to Class Exericse 6-1

1. No, Yes, $^{-}1$
2. N + 1 , N - 1
3. (a) Yes (b) No (c) No (d) No
4. (a) $\frac{23}{4}$ (b) $\frac{57}{8}$ (c) $\frac{12}{1}$ (d) $\frac{47}{100}$
   (e) $^{-}(\frac{5}{3})$ (f) $\frac{69}{10}$ (g) $\frac{37}{10}$ (h) $^{-}(\frac{20}{3})$
5. $\frac{7}{4}$
6. $^{-}(\frac{4}{7})$
7. Reciprocal
8. 1
9. If they are the same when written in simplest form. If the numerators are equal when the denominators are equal.
10. $\frac{10}{14}$, $\frac{15}{21}$, $\frac{20}{28}$, etc.

Answers to Exercises 6-1

1. (a) Closure under addition.
   (b) Identity for addition.
   (c) Identity for multiplication.

(d) Closure under multiplication.

(e) Distributive property of multiplication over addition.

(f) Commutative property of multiplication.

(g) Associative property of addition.

2. (a) $\frac{35}{2}$

(b) ${}^{-}\left(\frac{5}{7}\right)$

(c) ${}^{-}\left(\frac{11}{3}\right)$

(d) ${}^{-}\left(\frac{7}{20}\right)$

(e) $\frac{10}{1}$

(f) $\frac{1703}{100}$

3. (a) $\frac{1}{2}$

(b) $\frac{3}{4}$

(c) ${}^{-}\left(\frac{31}{50}\right)$

(d) $\frac{44}{7}$

(e) ${}^{-}\left(\frac{33}{4}\right)$

(f) $\frac{25}{2}$

4. (a) ${}^{+}28$ or 28

(b) ${}^{-}756$

(c) ${}^{-}\left(3\frac{1}{7}\right)$ or ${}^{-}\left(\frac{22}{7}\right)$

(d) ${}^{+}\frac{176}{5}$ or $\frac{176}{5}$

5. One.

6. Zero.

7. ${}^{-}4$, ${}^{-}\left(\frac{2}{3}\right)$, 0 , $\frac{3}{8}$, $\frac{2}{5}$, 0.41 , $\frac{7}{16}$, $\frac{4}{7}$

8. ${}^{-}2$.

9. No. The average of an odd integer and an even integer is never an integer.

$\frac{1}{2}({}^{-}3 + 8) = 2\frac{1}{2}$

10. (a) 3.3333

(b) 0.90909

(c) 163.1212

(d) 1.42142

(e) 134.6333

(f) 8464.646

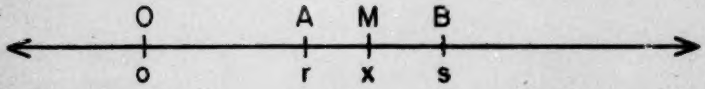
11. (a) 33.333
(b) 9.0909
(c) 1631.212
(d) 14.2142
(e) 1346.333
(f) 84646.46

12. (a) 333.33
(b) 90.909
(c) 16312.12
(d) 142.142
(e) 13463.33
(f) 846464.6

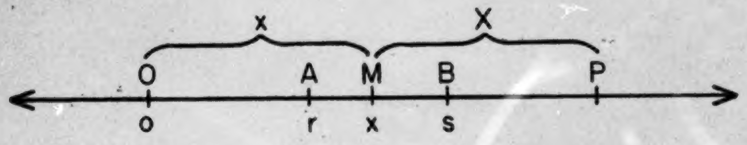
6-2. Density of Rational Numbers

The exercises on inserting more and more fractions in their proper order between 0 and 1 are intended to introduce the feeling of density without the precise statement of the property.

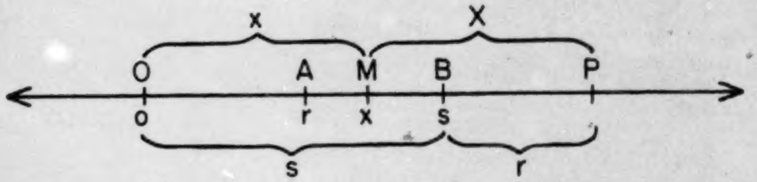
The development of the coordinate for the mid-point of a segment is incomplete. It was thought that a complete discussion might divert the attention of some of the pupils from the central idea of density. If you wish to fill in the gaps, you might go through the following argument with your class, perhaps using specific numbers for r and s. Consider two rational numbers r and s and the corresponding points A and B on the number line. Let M be the mid-point of the segment $\overline{AB}$.



Is M a rational point? If x is the number corresponding to M , is x rational? Let us investigate. Construct a point P to the right of M with $MP = x = OM$.



Since $OM = MP$ and $AM = MB$, we must have $OA = BP$. But $OA = r$, since A is the point corresponding to the number r . Thus $BP = r$.



From this we see two ways of finding OP . $OP = x + x = 2x$ and also $OP = r + s$. Thus $2x$ and $r + s$ are names for the same number, so

$$2x = r + s \quad .$$

If "twice x" is $r + s$ then x must be $\frac{1}{2}(r + s)$;

$$x = \frac{1}{2}(r + s) \quad .$$

Thus M is a rational point corresponding to the average of r and s .

Your class may be interested in discovering other ways of finding rational numbers between rational numbers. They may think of methods by themselves. Here are two which you might suggest if you wish.

A. Suppose $\frac{a}{b} < \frac{c}{d}$. Write $\frac{a}{b} = \frac{ap}{bp}$, $\frac{c}{d} = \frac{cq}{dq}$ where $bp = dq$ and bp is positive. (That is, write fractions with positive common denominators for the two numbers.) It is then easy to find a rational number between $\frac{a}{b}$ and $\frac{c}{d}$: If $ap < n < cq$, then $\frac{ap}{bp} < \frac{n}{bp} < \frac{cq}{dq}$. However, if $cq = ap + 1$ there is no such integer n . In this case multiply $\frac{ap}{bp}$ and $\frac{cq}{dq}$ by $\frac{2}{2}$ and proceed as before. For example, to find a rational number between $\frac{10}{11}$ and $\frac{11}{12}$ we can first multiply $\frac{10}{11} \times \frac{12}{12}$ and $\frac{11}{12} \times \frac{11}{11}$ obtaining $\frac{120}{132}$ and $\frac{121}{132}$. Since there is no integer between 120 and 121 we now multiply both numbers by $\frac{2}{2}$. Thus the two numbers will be $\frac{240}{264}$ and $\frac{242}{264}$. A number between them is now seen to be $\frac{241}{264}$.

B. Suppose $0 < \frac{a}{b} < \frac{c}{d}$, and a , b , c , d are counting numbers. If we write $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$ we have $\frac{ad}{bd} < \frac{bc}{bd}$. Since the denominators are equal we must have $ad < bc$. Using this same method, compare $\frac{a}{b}$ and $\frac{a+c}{b+d}$. We write

$$\frac{a}{b} = \frac{a(b+d)}{b(b+d)} \text{ and}$$

$$\frac{a+c}{b+d} = \frac{b(a+c)}{b(b+d)} .$$

The denominators are the same. What about the numerators? $a(b+d) = ab + ad$ and $b(a+c) = ab + bc$, by the distributive property. Since $ad < bc$ we have $ab + ad < ab + bc$ so

$$\frac{a}{b} < \frac{a+c}{b+d} .$$

Similarly we could show that

$$\frac{a+c}{b+d} < \frac{c}{d} ,$$

so $\frac{a+c}{b+d}$ is a rational number between $\frac{a}{b}$ and $\frac{c}{d}$. Caution: the pupil should not confuse $\frac{a+c}{b+d}$ with $\frac{a}{b} + \frac{c}{d}$.

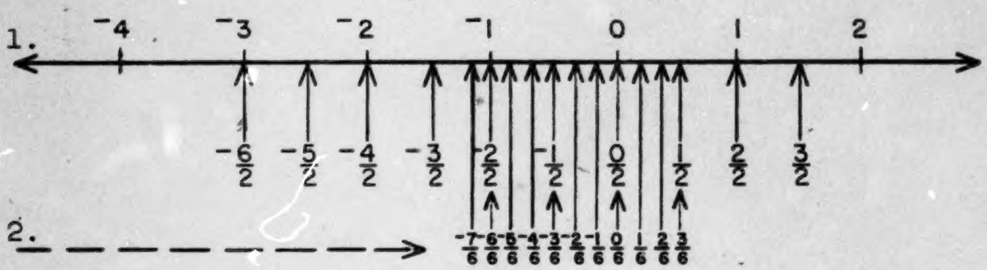
For example, given that $0 < \frac{2}{3} < \frac{7}{8}$.

We write $\frac{2}{3} = \frac{2}{3} \cdot \frac{8}{8} = \frac{16}{24}$ and $\frac{7}{8} \cdot \frac{3}{3} = \frac{21}{24}$, and see that $16 < 21$. Now let us look at $\frac{2+7}{3+8}$ or $\frac{9}{11}$ and compare it with $\frac{2}{3}$,

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{11}{11} = \frac{22}{33} \text{ and } \frac{9}{11} = \frac{9}{11} \cdot \frac{3}{3} = \frac{27}{33} .$$

So, $\frac{2}{3} < \frac{9}{11}$ and now we look at $\frac{9}{11}$ with respect to our other number $\frac{7}{8}$. We see that $\frac{9}{11} = \frac{9}{11} \cdot \frac{8}{8} = \frac{72}{88}$ and $\frac{7}{8} = \frac{7}{8} \cdot \frac{11}{11} = \frac{77}{88}$, so $\frac{9}{11} < \frac{7}{8}$ and therefore $\frac{2}{3} < \frac{9}{11} < \frac{7}{8}$. Another way then to find a rational between two rationals is to add the number in the numerator for the new numerator, and add the numbers in the denominator for the new denominator.

Answers to Exercises 6-2a



3. Yes. $\frac{^-6}{6}$ and $\frac{^-2}{2}$, $\frac{^-3}{6}$ and $\frac{^-1}{2}$, $\frac{0}{6}$ and $\frac{0}{2}$, $\frac{3}{6}$ and $\frac{1}{2}$.

4. Six new points. $\frac{8}{7}$, $\frac{9}{7}$, $\frac{10}{7}$, $\frac{11}{7}$, $\frac{12}{7}$, $\frac{13}{7}$.

   Six new points. $\frac{22}{7}$, $\frac{23}{7}$, $\frac{24}{7}$, $\frac{25}{7}$, $\frac{26}{7}$, $\frac{27}{7}$.

5. Four new points. The first integer plus $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$, $\frac{7}{8}$.

6. (a) $\frac{0}{3}$ names the same point as $\frac{0}{1}$.

(b) $\frac{2}{4}$ names the same point as $\frac{1}{2}$.

(c) In the row for sixths, $\frac{2}{6}$, $\frac{3}{6}$, and $\frac{4}{6}$ have already been named. Since 5 and 7 are prime numbers, points named by rational numbers with denominators 5 and 7 have not yet been named.

7. $\frac{1}{8} > 0$. $\frac{1}{8} = \frac{7}{56}$ and $\frac{1}{7} = \frac{8}{56}$, $\frac{8}{56} > \frac{7}{56}$.

$\frac{1}{7} = \frac{6}{42}$ and $\frac{1}{6} = \frac{7}{42}$, $\frac{7}{42} > \frac{6}{42}$.

$\frac{1}{6} = \frac{5}{30}$ and $\frac{1}{5} = \frac{6}{30}$, $\frac{6}{30} > \frac{5}{30}$.

$\frac{1}{5} = \frac{4}{20}$ and $\frac{1}{4} = \frac{5}{20}$, $\frac{5}{20} > \frac{4}{20}$.

$\frac{3}{4} = \frac{15}{20}$ and $\frac{4}{5} = \frac{16}{20}$, $\frac{16}{20} > \frac{15}{20}$.

$\frac{4}{5} = \frac{24}{30}$ and $\frac{5}{6} = \frac{25}{30}$, $\frac{25}{30} > \frac{24}{30}$.

$\frac{5}{6} = \frac{35}{42}$ and $\frac{6}{7} = \frac{36}{42}$, $\frac{36}{42} > \frac{35}{42}$.

$\frac{6}{7} = \frac{48}{56}$ and $\frac{7}{8} = \frac{49}{56}$, $\frac{49}{56} > \frac{48}{56}$.

$\frac{1}{1} = \frac{8}{8}$ and $\frac{8}{8} > \frac{7}{8}$.

8. Six. Four. Ten.

9. Eleven. Prime number. See answer to Problem 6 (c).

Answers to Exercises 6-2b

1. No. There is no integer between -5 and -4 .
2. Yes, 1 is the smallest. No largest positive integer.
3. No smallest negative integer. -1 is the largest.
4. No. No.
5. $\frac{1}{200}$. $\frac{1}{400}$. $\frac{3}{400}$.
6. $\frac{3}{40}$, $\frac{5}{80}$ or $\frac{1}{16}$, $\frac{7}{80}$
7. One such plan would be to use some other point than the halfway point. For example, the point $\frac{1}{3}$ of the way

from $\frac{1}{1000}$ to $\frac{2}{1000}$ is $\left(\frac{1}{1000}+\frac{1}{3000}\right)$ or $\frac{4}{3000}$. The next named points would be $\left(\frac{1}{1000}+\frac{1}{9000}\right)$ or $\frac{10}{9000}$; $\left(\frac{1}{1000}+\frac{1}{27,000}\right)$ or $\frac{28}{27,000}$; $\left(\frac{1}{1000}+\frac{1}{81,000}\right)$ or $\frac{82}{81,000}$; $\left(\frac{1}{1000}+\frac{1}{243,000}\right)$ or $\frac{244}{243,000}$.

6-3. Decimal Representations for Rationals

Note that we are considering a terminating decimal such as 3.75 as a periodic decimal $3.75\overline{0}\ldots$. Thus we are not considering two kinds of decimals, terminating and periodic, but only periodic (infinite) decimals. This enables us to say, conveniently, that every rational number has a periodic decimal representation. Point this out to your class. You might say that we call such decimals as $3.75\overline{0}\ldots$ terminating not because they "end," but because the ordinary process of long division which we might use to find this decimal terminates:

```
   3.75
4)15.00
  12
   30
   28
    20
    20
     0
```

We would carry this calculation no further. There is no need for it.

Answers to Exercises 6-3

1. (a) 2.2500000000...
   (b) 0.2083333333...
   (c) 0.0428571428...
   (d) 0.0857142857...
   (e) 0.2439024390...
   (f) 1.0240000000...
   (g) 0.0378378378...
   (h) 0.0121012101...
   (i) 0.0121951219...
   (j) 0.0588235294117647O5...

2. a, c, d, g, i.

3. (a) 2 (g) $2\cdot2\cdot2$
   (c) $2\cdot2$ (i) $2\cdot5$
   (d) 5

4. (a) 0.142857 (d) 0.571428
   (b) 0.285714 (e) 0.714285
   (c) 0.428571 (f) 0.857142

6-4 The Rational Number Corresponding to a Periodic Decimal

After reading this section and working the exercises, the pupil should understand that to find a fraction for the number n, represented by a repeating decimal he multiplies the number by 10^k, where k is the number of digits in the repeating group, and subtracts the original number from this product. The difference will always be a terminating decimal which will then be equal to $(10^k - 1)n$. After obtaining n as

$$\frac{(\text{terminating decimal})}{10^k - 1}$$

it may be necessary to multiply numerator and denominator by a power of 10 in order to obtain a fraction for n of the form $\frac{p}{q}$, p and q integers.

For example, to find the fraction for the decimal $n = 31.7257\overline{57}\ldots$ we multiply by 10^2 because there are two digits in the repeating group. Then we subtract n from $100n$,

$$\begin{aligned} 100n &= 3172.57\overline{57}\ldots \\ n &= 31.72\overline{57}\ldots \\ \hline 99n &= 3140.85 \\ \text{and} \quad n &= \frac{3140.85}{99}. \end{aligned}$$

To change this to the form $\frac{p}{q}$ where p and q are integers it is now necessary to multiply by $\frac{10^2}{10^2}$, obtaining $\frac{314085}{9900}$. In this case it is also possible to factor and "reduce,"

$$\frac{3\cdot 5\cdot 20939}{2\cdot 2\cdot 3\cdot 3\cdot 5\cdot 5\cdot 11} = \frac{20939}{660}$$

This is in simplist form since none of the prime factors of 660 (2 , 3 , 5 , 11) are factors of 20939.

Answers to Class Exercise 6-4

1. (a) 9n (d) 90n
(b) 99n (e) 999n
(c) 990n (f) 9,900n

2. (a) $9.9\overline{9}\ldots$ (f) $613.45\overline{345}\ldots$
(b) $312.12\overline{12}\ldots$ (g) $803.15\overline{15}\ldots$
(c) $35.035\overline{035}\ldots$ (h) $31289.9\overline{9}\ldots$
(d) $166.6\overline{6}\ldots$ (i) $3128.9\overline{9}\ldots$
(e) $0.04\overline{4}\ldots$ (j) $60123.0123\overline{0123}\ldots$

3. (a) $2816.10\overline{0}\ldots$ or 2816.1 (e) $1.110\overline{0}\ldots$ or 1.1
(b) $9.0\overline{0}\ldots$ or 9 (f) $351.0\overline{0}\ldots$ or 351
(c) $162.0\overline{0}\ldots$ or 162 (g) $27048.\overline{00}\ldots$ or 27048
(d) $298.0\overline{0}\ldots$ or 298 (h) $374.830\overline{0}\ldots$ or 374.83

4. (a) $\frac{31}{990}$ (d) $\frac{103}{99900}$
(b) $\frac{411}{900}$ (e) $\frac{3824}{90}$
(c) $\frac{163}{990}$ (f) $\frac{47531}{9999000}$

5. (a)
$$\begin{aligned} 10n &= 5.5\overline{5}\ldots \\ \underline{\quad n} &\underline{= 0.5\overline{5}\ldots} \\ 10n - n &= 5.0\overline{0}\ldots \end{aligned}$$

(b)
$$\begin{aligned} 100n &= 73.73\overline{73}\ldots \\ n &= 0.73\overline{73}\ldots \\ \hline 100n - n &= 73.0\overline{0}\ldots \end{aligned}$$

(c)
$$\begin{aligned} 1000n &= 901.901\overline{901}\ldots \\ n &= 0.901\overline{901}\ldots \\ \hline 1000n - n &= 901.0\overline{0}\ldots \end{aligned}$$

(d)
$$\begin{aligned} 10n &= 30.233\overline{3}\ldots \\ n &= 3.023\overline{3}\ldots \\ \hline 10n - n &= 27.210\overline{0}\ldots \end{aligned}$$

(e)
$$\begin{aligned} 10n &= 1{,}631.77\overline{7}\ldots \\ n &= 163.17\overline{7}\ldots \\ \hline 10n - n &= 1{,}468.60\overline{0}\ldots \end{aligned}$$

(f)
$$\begin{aligned} 100n &= 67{,}242.42\overline{42}\ldots \\ n &= 672.42\overline{42}\ldots \\ \hline 100n - n &= 66{,}570.0\overline{0}\ldots \end{aligned}$$

(g)
$$\begin{aligned} 100n &= 12.3456\overline{56}\ldots \\ n &= 0.1234\overline{56}\ldots \\ \hline 100n - n &= 12.22220\overline{0}\ldots \end{aligned}$$

(h)
$$\begin{aligned} 10n &= 34.100\overline{0}\ldots \\ n &= 3.410\overline{0}\ldots \\ \hline 10n - n &= 30.690\overline{0}\ldots \end{aligned}$$

Answers to Exercises 6-4

1. (a) $\frac{1}{11}$
   (b) $\frac{1}{9}$
   (c) $\frac{1}{18}$
   (d) $\frac{41}{333}$
   (e) $\frac{1625}{10000} = \frac{65}{400} = \frac{13}{80}$
   (f) $\frac{1}{6}$
   (g) $\frac{5120}{999}$
   (h) $\frac{221}{22}$
2. (a) $32 = 2\cdot2\cdot2\cdot2\cdot2$ or 2^5
   (b) $100 = 2\cdot2\cdot5\cdot5$ or $2^2\cdot5^2$

(c) $9 = 3 \cdot 3$ or 3^2

(d) $50 = 2 \cdot 5 \cdot 5$ or $2 \cdot 5^2$

(e) $35 = 5 \cdot 7$

(f) $80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$ or $2^4 \cdot 5$

(g) $120 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5$ or $2^3 \cdot 3 \cdot 5$

(h) $160 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5$ or $2^5 \cdot 5$

3. The numbers in Parts a, b, d, f, h.

4. 64 and 80.

6-5. Rational Points on the Number Line

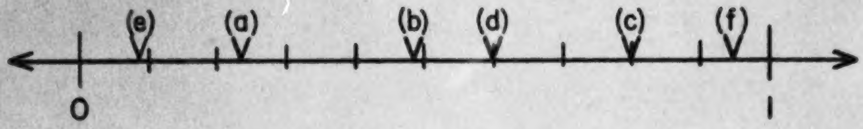
The important understanding which the pupil should acquire in reading this section is that each repeating decimal corresponds to a unique point on the number line. We can locate the point with more and more accuracy by using finer and finer subdivisions on the number line and simultaneously looking at more and more of the digits in the decimal. Emphasize that there is only one point corresponding to the entire decimal and that by locating P with more and more accuracy, we are only getting closer and closer to that point.

Answers to Exercises 6-5

1. (a) 1.372 1.379 1.385 1.493 5.468
   (b) $^{-}9.426$ $^{-}5.630$ $^{-}2.765$ $^{-}2.763$ $^{-}2.761$
   (c) $^{-}0.15475$ 0.15463 0.15467 0.15475 0.15598

2. (a) none
   (b) all but $^{-}0.15475$
   (c) all but $^{-}0.15475$
   (d) all but $^{-}0.15475$
   (e) 0.15475 0.15467 0.15463

3.



4. (a) $\frac{17}{50} = 0.340\overline{0}\ldots$, $\frac{3}{9} = 0.3\overline{3}\ldots$, $\frac{4}{10} = 0.40\overline{0}\ldots$

(b) $\frac{2}{3} = 0.6\overline{6}\ldots$, $\frac{67}{100} = 0.670\overline{0}\ldots$, $\frac{7}{10} = 0.70\overline{0}\ldots$

(c) $\frac{3}{7} = 0.\overline{428571}\ldots$, $\frac{4}{9} = 0.4\overline{4}\ldots$

(d) $\frac{152}{333} = 0.456\overline{456}\ldots$, $\frac{415}{909} = 0.4565\overline{4565}\ldots$

6-6. Irrational Numbers

This section contains one of the very few formal proofs in this Volume. The whole purpose of the chapter may be defeated if the pupil is allowed to become mired in the details. The ideas involved in the proof are elementary, and an eighth grader should be able to follow the line of reasoning. However, it is far more important that the pupil be convinced by the proof of the irrationality of $\sqrt{2}$ than that he master the proof to the point of being able to reproduce it. Naturally, your attitude as a teacher will be an important factor in shaping the pupil's reaction. You should be thoroughly acquainted with the details of the proof and treat it as a natural line of discussion, rather than making a "big thing" of it.

If the statement, "if a^2 is even then a is even," troubles the pupil, the following explanation may be helpful:

> Remember the unique factorization property: A counting number may be factored as a product of primes in only one way (except for the order of the factors). The prime factors of a^2 are just the prime factors of a, each one taken twice in the product.
>
> If 2 is one of the primes in the factorization of a^2 (that is, if a^2 is even) then it can be there only by

being one of the prime factors of a (and there must then be *two* factors 2 in a^2). Thus a must be even, too.

This same reasoning shows that if a prime p divides a^2 then p must also divide a . This is useful in proving that $\sqrt{p}$ is irrational if p is a prime. The case $p = 5$ is an exercise for the pupil (Problem 8, Exercises 6-6.).

You may wish to extend the discussion of the method of proof which we call "indirect reasoning." Many examples of indirect reasoning can be found in everyday life. What parent has never heard, "*If you loved me*, you would let me . . . !" This is a fragment of an indirect proof of the "fact" that the parent does not love the child. (Fortunately, such "facts" usually dry up with the tears.) "If you loved me" is the assumption of the opposite of the statement the child obviously wishes to prove at the moment. The conclusion, "you would let me ...," apparently contradicts something which has just become an *established* fact. The assumed statement "You love me," must, therefore, be very false.

A less facetious example is the proof of the statement:

The number 0 has no inverse under multiplication.

Suppose we assume that 0 has an inverse b . Then, by definition of a multiplicative inverse, $0 \cdot b = 1$. This is a direct contradiction of the facts that $0 \cdot x = 0$, for all numbers x , and $0 \neq 1$. Thus our assumption that 0 has an inverse must be false.

The classical name for this type of reasoning is *reductio ad absurdum*. It is still common to refer to a "reductio proof."

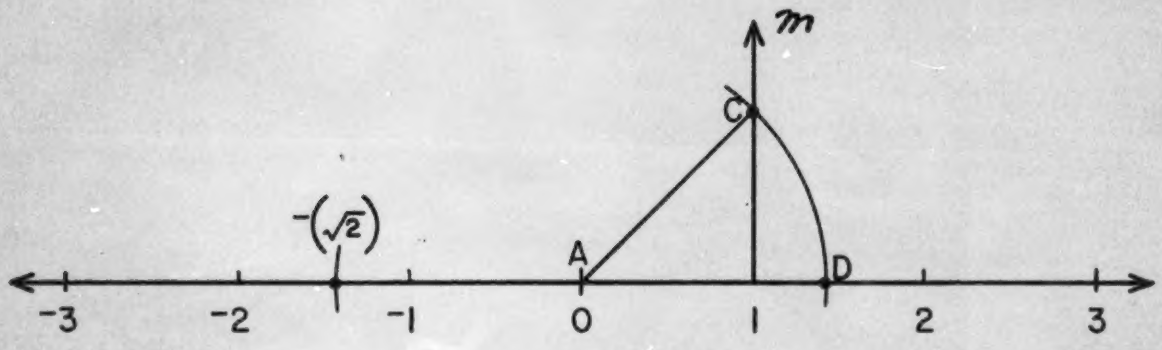
Another excellent example of a reductio proof is contained in the reference to Gamow's *One, Two, Three,. . . Infinity* on the question of the non-enumerability of the set of irrational numbers. One actually proves that the set of *real* numbers between

0 and 1 cannot be enumerated, by assuming that they can be enumerated by some scheme and then constructing a decimal which has been missed by the enumeration. The assumption that you have enumerated the real numbers between 0 and 1 leads to the conclusion that you have not.

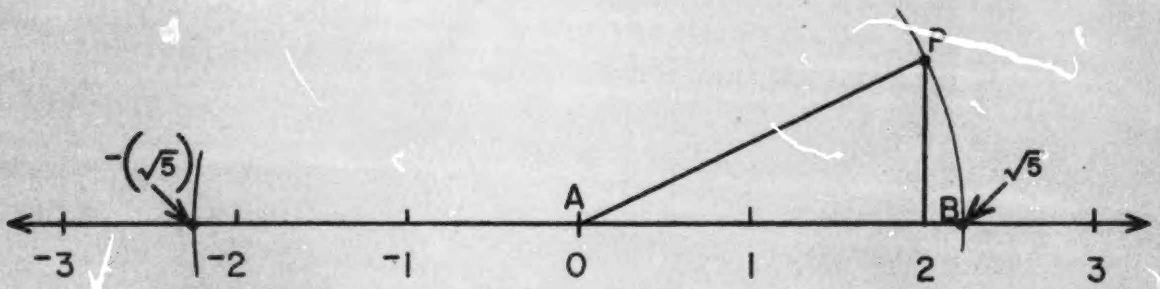
The fact that the set of real numbers is not enumerable, while the set of rational numbers is, implies that the irrational numbers themselves form a set which is not enumerable.

Answers to Exercises 6-6

1.



2.

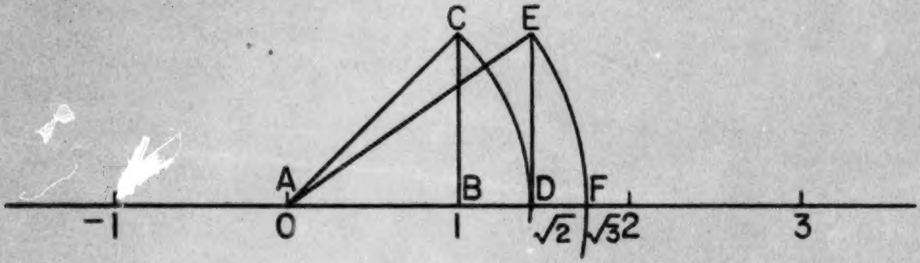


The measure of $\overline{AP}$ is $\sqrt{5}$.

3. See diagram for Problem 2.

4. Irrational. There is no rational number $\frac{a}{b}$ such that $(\frac{a}{b})^2 = 5$.

*5.



AB = 1 , BC = 1 , So AC = $\sqrt{2}$. Make AD = AC , so point D represents $\sqrt{2}$.

Draw DE perpendicular to AD , DE = 1 . Then $AE^2 = (\sqrt{2})^2 + 1^2$ and AE = $\sqrt{3}$. Make AF = AE so point E corresponds to $\sqrt{3}$.

Similarly $\sqrt{6}$ can be found by using a right triangle with sides of lengths $\sqrt{5}$ and 1; $\sqrt{7}$ by using a right triangle with sides of lengths $\sqrt{6}$ and 1 .

6. Lay off the distance $\sqrt{2}$ twice to the right of 0 ; 3 times to the right of 0 ; three times to the left of 0 .

7. Irrational.

*8. BRAINBUSTER: First show the following:

I. If a number is divisible by 5 , it can be expressed as $5 \cdot k$ where k is an integer.

II. If a number is not divisible by 5 , it can be expressed as $5n + R$, with $0 < R < 5$, R an integer. (Division Algorithm, Volume 1, Chapter 5)

III. If n is divisible by 5, then n^2 is divisible by 5.

$[(5k)^2 = 25k^2 = 5(5k^2)$. $5k^2$ is an integer by the closure property of integers under multiplication.]

IV. If n is not divisible by 5, then n^2 is not divisible by 5. If n is not divisible by 5, it may be expressed as $5k + 1$, or $5k + 2$, or $5k + 3$, or $5k + 4$.

$(5k + 1)^2 = 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1$. $\therefore$ it is not divisible by 5.

$(5k + 2)^2 = 25k^2 + 20k + 4 = 5(5k^2 + 4k) + 4$.

$\therefore$ it is not divisible by 5.

$(5k + 3)^2 = 25k^2 + 30k + 9 = 5(5k^2 + 6k + 1) + 4$.

$\therefore$ it is not divisible by 5.

$(5k + 4)^2 = 25k^2 + 40k + 16 = 5(5k^2 + 8k + 3) + 1$.

$\therefore$ it is not divisible by 5.

V. If n is divisible by 5, and $\sqrt{n}$ is an integer, then $\sqrt{n}$ is divisible by 5.

Use an indirect argument.

Assume $\sqrt{n}$ is not divisible by 5. Then by (4) above, n is not divisible by 5. But n is divisible by 5. Therefore $\sqrt{n}$ is divisible by 5.

Now the proof follows the same form as the proof that 2 is not a rational number.

1. Assume $\sqrt{5}$ is a rational number. Then $\sqrt{5} = \frac{p}{q}$, p and q integers and $q \neq 0$. Let $\frac{p}{q}$ be the simplest expression for $\sqrt{5}$, so p and q have no common factors other than 1.
2. $5 = \frac{p^2}{q^2}$.
3. $\therefore 5q^2 = p^2$.
4. q^2 is an integer.

5. $\therefore p^2$ is divisible by 5 .

6. $\therefore p$ is divisible by 5 .

Let $p = 5k$, k an integer.

7. $5q^2 = (5k)^2$ or $25k^2$.

8. $q^2 = 5k^2$.

9. k^2 is an integer .

10. $\therefore q^2$ is divisible by 5 .

11. $\therefore q$ is divisible by 5 .

12. By statements 6 and 11 , both p and q have the factor 5 . But this contradicts the condition in Statement 1. Therefore the assumption that $\sqrt{5}$ is a rational number is false.

13. $\therefore$ $\sqrt{5}$ is not a rational number.

Answers to Exercises 6-7

1. (a) $5 < \sqrt{30} < 6$.
   (b) $9 < \sqrt{89} < 10$.
   (c) $15 < \sqrt{253} < 16$.
   (d) $65 < \sqrt{4280} < 66$.
   (e) $96 < \sqrt{9315} < 97$.

2. (a) 3
   (b) 2.999824
   (c) 3.003289
   (d) .000176 .003289
   (e) 1.732

3. $(1.73)^2 = 2.9929$ $(1.74)^2 = 3.0276$ 1.73 is the better

4. $(3.87)^2 = 14.9769$ $(3.88)^2 = 15.0544$ 3.87 is the better

5. $(25.2)^2 = 635.04$ $(25.3)^2 = 640.09$ 25.2 is the better

6. 3.2

7. 12.2

8. 14.9

9. $n \approx 3.2$

10. $n \approx 12.2$

6-8. Irrational Numbers and the Real Number System

When they have read sections 7 and 8, you might point out to your class that in obtaining a decimal representation for $\sqrt{2}$ we have not used any special properties of the number 2 other than the fact that it is a positive number and corresponds to a point on the number line. Any positive number N - rational or irrational - could be used in place of 2. Following the procedure of section 7, we would obtain a sequence of decimals:

$$b_1b_2 \ldots b_k.a_1$$
$$b_1b_2 \ldots b_k.a_1a_2$$
$$b_1b_2 \ldots b_k.a_1a_2a_3$$
$$b_1b_2 \ldots b_k.a_1a_2a_3a_4$$
$$\cdots\cdots\cdots$$

whose squares become closer and closer to N. The infinite decimal $b_1b_2 \ldots b_k.a_12_22_32_4 \ldots$ which is obtained would represent a real number whose square is N. Thus, the real number system has a property which we have not stated in the book: Every positive real number has a square root which is also a real number. A similar statement holds for arbitrary nth roots, where n is a counting number.

Answers to Exercises 6-8

1.

Rational	Irrational
(a) $0.231\overline{231}\ldots$	(b) $0.23123112311123\ldots$
(c) $\frac{3\sqrt{2}}{7\sqrt{2}} = \frac{3}{7}$	(d) $\sqrt{7}$
(e) $0.78\overline{342}\ldots$	(f) $\frac{\pi}{2}$
(i) $0.750\overline{0}\ldots$	(g) $\frac{3}{4}\sqrt{6}$
(j) $\frac{58}{11}$	(h) $9 - \sqrt{3}$
	(k) $0.959559555955559\ldots$

Note that in (b) it is intended that the number of ones continues to increase.

Similarly, in (k) it is intended that the number of fives continues to increase.

2. (a) $0.231\overline{231}\ldots = \frac{231}{999} = \frac{77}{333}$

(c) $\frac{3\sqrt{2}}{7\sqrt{2}} = \frac{3}{7} = 0.428571\overline{428571}\ldots$

(e) $0.78\overline{342}\ldots = \frac{78264}{99900} = \frac{2174}{2775}$

(i) $0.750\overline{0}\ldots = \frac{3}{4}$

(j) $\frac{58}{11} = 5.27\overline{27}\ldots$

3.

	Irrational Number	Nearest Hundredth
(b)	$0.23123112311123\ldots$	0.23
(d)	$\sqrt{7}$	2.65
(f)	$\frac{\pi}{2}$	1.57
(g)	$\frac{3}{4}\sqrt{6}$	1.84
(h)	$9- \sqrt{3}$	7.27
(k)	$0.959559555955559\ldots$	0.96

Some possible answers for 4 , 5 , 6 , 7 .

4. (a) $.45632\overline{0}\ldots$ $8.27\overline{0}\ldots$ $.75\overline{0}\ldots$

(b) $.45\overline{45}\ldots$ $12.762\overline{762}\ldots$ $.063\overline{63}\ldots$

(c) .450450045... .123112311123... .565665666...

5. (a) $2.375\overline{375}\ldots$ (b) 2.370370037000...

6. (a) $.34\overline{4}\ldots$ (b) .34563445634456...

7. (a) 67.282... (b) 67.28292020020002...

8. No. There are no square roots of the negative integers in the real number system.

9. $\frac{355}{113} = 3.1415929\ldots$

$\frac{22}{7} = 3.1428571\ldots$

$\pi = 3.1415926\ldots$

$\frac{355}{113}$ is a much better approximation to π than is $\frac{22}{7}$.

6-9. Rational Approximations to Irrationals, and

6-10. Rationals and Irrationals in the World Around Us

These two sections are designed to be one day's work. Thus, there are no exercises following section 9.

Answers to Exercises 6-10

1. (a) irrational - circumference is π units.

(b) rational - area is 1 square unit.

(c) rational - hypotenuse is 13 units.

(d) rational - $\left(\sqrt{3}\right)^2 = 3$.

(e) irrational - volume is 2π cubic units.

(f) rational - (each side is $\sqrt{2}$ units).

2. $(1.414\ldots) \times (1.732\ldots) \approx 1.414 \times 1.732 \approx 2.449$

3. (a) $\sqrt{2} = (1.414\ldots)$ $\frac{1}{2}\sqrt{2} = (0.707\ldots)$

$\sqrt{2} \times \frac{1}{2}\sqrt{2} = (1.414\ldots) \times (0.707\ldots)$

$\approx 1.414 \times 0.707 = 0.999698$

$\sqrt{3} + \sqrt{2} = (1.732\ldots) + (1.414\ldots) \approx 3.146$

$\sqrt{3} - \sqrt{2} = (1.732\ldots) - (1.414\ldots) \approx 0.318$

$(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} - \sqrt{2}) \approx (3.146) \times (0.318)$

$= 1.000428$

(b) $\sqrt{2} \times \frac{1}{2}\sqrt{2} = \frac{1}{2}(\sqrt{2})^2 = \frac{1}{2} \cdot 2 = 1$

$(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} - \sqrt{2}) = (\sqrt{3} + \sqrt{2}) \cdot (\sqrt{3} + {}^{-}\sqrt{2})$

$= (\sqrt{3} + \sqrt{2}) \cdot \sqrt{3} + (\sqrt{3} + \sqrt{2}) \cdot {}^{-}\sqrt{2}$

$= (\sqrt{3})^2 = (\sqrt{2} \cdot \sqrt{3}) + {}^{-}(\sqrt{3} \cdot \sqrt{2}) + {}^{-}(\sqrt{2})^2$

$= 3 + 0 + {}^{-}2$

$= 1$

4. Using 3.1416 for π the radius $\approx \frac{1}{3.1416}$

$\approx 0.318309\ldots$

Chapter 6

SAMPLE QUESTIONS

1. Fill in each blank with either true or false.

	Counting Numbers	Integers	Rationals	Real Numbers
Closure for addition				
Closure for multiplication				
Closure for subtraction				
Closure for division				
There is always an additive inverse				
There is always a reciprocal (Except for zero)				
There is density.				

2. Find two numbers between $\frac{10}{11}$ and $\frac{11}{12}$.

3. What rational number is represented by $0.63\overline{63}$?

4. What is the decimal name for $\frac{5}{80}$?

5. Which of these have decimals that repeat zero,
$\frac{5}{6}$, $\frac{5}{8}$, $\frac{5}{9}$, $\frac{5}{15}$?

6. Write the closest integer to $\frac{35}{16}$ that is smaller than $\frac{35}{16}$.

7. To change the following to a fraction, multiply by (circle one): 10 , 100 , 1,000 .

$670.3131\overline{31}...$

True - False (8-17)

8. Every real number can be written as a rational number.
9. The smallest positive integer is one.
10. $\sqrt{2}$ is a number which when squared is equal to 2 .
11. Three and one-seventh is a rational number.
12. Every repeating decimal is a rational number.
13. The square root of seven is 1.645 .
14. Every real number can be represented by a point on the number line.
15. The number zero is not a rational number.
16. There are 12 integers between 15 and 27 .
17. A rational number may be expressed as an integer divided by a counting number.

Multiple Choice (18-21)

18. Which of the following is between $\frac{4}{9}$ and $\frac{2}{5}$?

    (a) $\frac{2}{3}$ (b) $\frac{14}{16}$ (c) $\frac{15}{16}$ (d) $\frac{19}{45}$ (e) $\frac{1}{3}$

19. Which of the following is equal to $\frac{5}{8}$?

    (a) 0.63 (b) 0.750 (c) 0.625 (d) 0.562

    (e) 0.580

20. Which of the following is not a rational number?

    (a) $\frac{12}{5}$ (b) $\frac{5}{0}$ (c) $\frac{7}{2}$ (d) 55 (e) $\frac{27}{64}$

21. Which of the following is equal to $0.123\overline{123}\ldots$?

    (a) $\frac{1}{11}$ (b) $\frac{1}{23}$ (c) $\frac{41}{333}$ (d) $\frac{3}{55}$ (e) $\frac{123}{99}$

22. Write an irrational number in decimal form.
23. Write a rational number between $0.625\overline{46}$ and $\frac{5}{8}$.

24. Which property is illustrated in the following sentence?

$$\sqrt{5}\,(2 + 7) = (\sqrt{5} \cdot 2) + (\sqrt{5} \cdot 7)$$

25. Draw a line segment that has a measure in inches of $\sqrt{6}$.

Answers to Sample Questions

1.	True	True	True	True
	True	True	True	True
	False	True	True	True
	False	False	False	False
	False	True	True	True
	False	False	True	True
	False	False	True	True

2. Some examples are $\frac{241}{264}$, $\frac{361}{396}$, $\frac{362}{396}$, $\frac{481}{528}$.

3. $\frac{7}{11}$

4. 0.0625

5. $\frac{5}{8}$

6. 2

7. 100

8. False

9. True

10. True

11. True

12. True

13. False

14. True

15. False

16. False

17. True

18. (d)

19. (c)

20. (b)

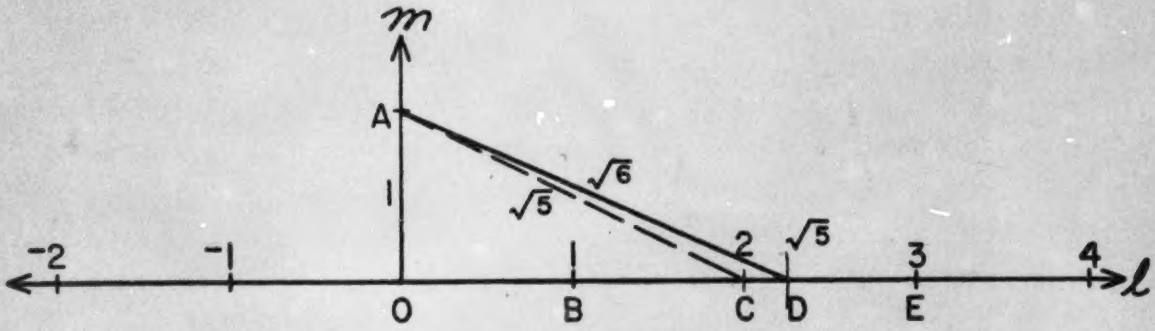
21. (c)

22. One example is 3.747747774...

23. One example is $0.6253\overline{0}$

24. Distributive property.

25. Here is a method slightly different from the one in the text.



At zero construct line m perpendicular to ℓ .
With a compass measure $\overline{AC}$, whose measure is $\sqrt{5}$.
Mark off on line ℓ , with compass point at zero ,
this distance of $\sqrt{5}$. Label this point D .
Draw $\overline{AD}$. This is the required segment whose measure
is $\sqrt{6}$.

Chapter 7

PERMUTATIONS AND SELECTIONS

General Remarks

The study of permutations and selections (or combinations) is a "fresh start" for many students because nearly all that is required as a background is a working knowledge of the four operations with numbers and an understanding of formulas. It might be well to call this fact to the attention of certain pupils who may be able but not always willing.

Committee arrangements are used in the beginning because the familiarity of pupils with this type of situation ensures that the only new ideas introduced are those of a purely mathematical nature. Other varied and interesting applications of the mathematical principles learned in this chapter are found in the exercises. The authors prefer to use the word "selection" rather than "combination" (although both words are found in the text). It is believed that "selection" would be a more natural and meaningful term for those working with these ideas for the first time.

Pascal's triangle, as introduced in Section 1, not only is an exciting application of selections but also serves other purposes as well. It is an excellent device for helping students recall quickly the number of selections that can be made from n things r at a time. Having this information about selections readily available speeds up class discussion and allows for development of concepts without interrupting the train of thought with constant, and sometimes irrelevant, calculations. Also the patterns involved in the triangle suggest certain inductive conclusions about the number relations among the various entries.

Contrary to what is probably the usual procedure, combinations (selections) are used as an introduction to permutations. By ordering the members of a subcommittee and applying the principles learned in the study of selections, students are prepared to accept and understand the following property: If n is a counting number, then the number of permutations of any set of n numbers is $n!$. The study of permutations in this chapter does not include those groups in which only a part of the items are repeated such as the permutations of the letters of the word, "Mississippi."

Some sections contain a list of Class Discussion Exercises which are planned for developmental class discussions and are not assignment exercises. The intention is not to limit discussion to the specific questions listed. It is expected that the teacher will use this list of questions merely as a guide. However, each question is carefully placed for developmental purposes in the order given and is pointed toward a particular idea in the developmental sequence. If this material is relatively unfamiliar to the teacher it is recommended that the sequence be adhered to as given. The exercises at the end of each part of a section are thought to be sufficient for one assignment.

It is hoped that the study of this chapter will widen pupils' appreciation for mathematics as a science. The investigation of Pascal's triangle begins with a nearly complete body of facts, includes the discovery and observation of quantitative relations, inductively suggests certain principles, verifies these deductively and derives generalizations. It is thought that the choice of Pascal's triangle might be a good one because the body of facts is small, the quantitative relations are of a very simple nature, the discovery of the patterns involved is not difficult, and the number of experiments that may be performed is ample.

It is also hoped that the study of this chapter will prove to be a source of intellectual satisfaction and fun for the pupil and will help to strengthen his realization of the power of mathematics. He may be impressed by the vast amount of hard work that would be required to determine the third term in the two-thousandth line of Pascal's triangle if he were actually to write down all the numbers in all the rows. He should be equally impressed by the ease with which the same information may be obtained by using $\frac{n(n-1)}{2}$.

For better students, reference may be made to Section 1 and to Problem 6, Exercise 3b of the supplementary unit on "Finite Differences."

A time allowance of 13 class periods is suggested for this chapter.

7-1 The Pascal Triangle

In this section, as in those to follow, the ideas are developed through carefully selected class discussion exercises. The questions in these exercises should be answered by students as they manipulate objects or write symbols for the elements of a set. Tabulating and counting are important key notions to be used throughout the section. Many students will prefer to use semi-concrete representations provided by writing symbols on paper but some use might also be made of concrete objects to be grouped and counted.

Perhaps the title for this section will not be very useful until the study of the section has been completed. If questions are asked about the title no harm will be done in displaying a triangle of three or four rows during the first discussion session. Some of the interesting properties of Pascal's triangle might even be pointed out at that time.

One of the important ideas to be brought out in this study is that if 3 objects have been selected from 5 then a selection of 2 objects has also been made. The symmetry of Pascal's triangle is based on this property. The student should clearly understand that the number of ways of selecting 17 objects from a set of 26 objects is exactly the same as the number of ways of selecting 9 objects from a set of 26 objects.

Students should be encouraged to observe that each entry in Pascal's triangle is the sum of the two numbers nearest to it on the preceding line. The better students, by the end of the chapter at least, may be able to state why this property is true. One way of showing this property follows. Consider 5 lines of Pascal's Triangle.

$$\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

Why is the number of combinations of 5 things taken three at a time the sum of the number of combinations of 4 things taken 2 at a time and the number of combinations of 4 things taken 3 at a time? Think of 5 letters A, B, C, D, E. If we select 3 of the 4 letters B, C, D, E then of course, we have part of the sets of 3 of the 5 letters A, B, C, D, E. The number we have is $\binom{4}{3}$. The ones we have not counted are the groups of 3 of which A is one letter of the group. Any group of 3 including A consists of A and 2 of the 4 letters B, C, D, E. The number of groups of 2 from four letters is $\binom{4}{2}$. Hence,

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}.$$

Or in more general terms, let us try to show

$$\binom{n}{3} = \binom{n-1}{3} + \binom{n-1}{2}.$$

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} \qquad \binom{n-1}{3} = \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}$$

$$\binom{n-1}{2} = \frac{(n-1)(n-2)}{1\cdot 2}$$

$$\binom{n-1}{3} + \binom{n-1}{2} = \frac{(n - 1)(n - 2)(n - 3)}{1\cdot 2\cdot 3} + \frac{(n - 1)(n - 2)}{1\cdot 2}\cdot\frac{3}{3}$$

$$= \frac{(n - 1)(n - 2)(n - 3 + 3)}{1\cdot 2\cdot 3}$$

$$= \frac{(n - 1)(n - 2)n}{1\cdot 2\cdot 3}$$

But $$\binom{n}{3} = \frac{n(n - 1)(n - 2)}{1\cdot 2\cdot 3}.$$

We have proved only a special case. Other special cases could be proved similarly. In order to prove the general property

$$\binom{n}{r} = \binom{n - 1}{r} + \binom{n - 1}{r - 1}$$

we would have to use mathematical induction.

The better students may also be interested in other properties of Pascal's Triangle. The sum of the numbers in any row is always a power of 2; the greatest common factor in any prime-numbered row is the number of the row; the third term in each row can be predicted by using the formula, $\frac{n(n - 1)}{2}$, and the fourth term in each row can be predicted by using the formula, $\frac{n(n - 1)(n - 2)}{1\cdot 2\cdot 3}$. Reference will be made to Pascal's Triangle in the sections to follow and even though no student observes these properties now, several may do so as the work progresses.

Particularly in this first section the treatment is intended to be quite informal. It would be better that no formulas be introduced into the general class discussions. However, the statement of generalizations which can be illustrated in special cases should be encouraged.

Answers to Class Discussion Exercises 7-1a

1. {A, D, E}.
2. Yes
3. {B, A, E}; {E, B, A}.
4. {ABC}, {ABD}, {ABE}, {ACD}, {ACE}, {BCD}, {BCE}, {CDE}, {DEA}, {DEB}.

5. 10

6. 6

7. B and D are on the same number of committees.

8. 4

9. $\frac{6}{10}$ or $\frac{3}{5}$

10. 3

11. $\frac{3}{6}$ or $\frac{1}{2}$

12. {B, C, D}

13. A and D.

14. 10

15. 6

16. 4

17. {C,A}; {C,B}; {C,D}; {C,E}.

Answers to Exercises 7-1a

1. (a) {A,B,C,D}, {A,B,C,E}, {A,B,D,E}, {A,C,D,E}, {B,C,D,E}.
   (b) 1
   (c) {A}, {B}, {C}, {D}, {E}.
   (d) Same.
   (e) 1
2. (a) 1
   (b) 4
   (c) {K}, {L}, {M}, {N}.
   (d) The three-member committee is composed of all the club members who are not in the corresponding one-member committee.

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13. (a) 1

(b) 0

(c) $\frac{1}{4}$

(d) $\frac{1}{2}$

14. List the possible lengths for a triangle:

10, 9, 8
10, 9, 7
10, 9, 6
10, 9, 5
10, 9, 4
10, 9, 3
10, 9, 2

10, 8, 7
10, 8, 6
10, 8, 5
10, 8, 4
10, 8, 3

10, 7, 6
10, 7, 5
10, 7, 4

10, 6, 5

9, 8, 7
9, 8, 6
9, 8, 5
9, 8, 4
9, 8, 3
9, 8, 2

9, 7, 6
9, 7, 5
9, 7, 4
9, 7, 3

9, 6, 5
9, 6, 4

8, 7, 6
8, 7, 5
8, 7, 4
8, 7, 3
8, 7, 2

8, 6, 5
8, 6, 4
8, 6, 3

8, 5, 4

7, 6, 5
7, 6, 4
7, 6, 3
7, 6, 2

7, 5, 4
7, 5, 3

6, 5, 4
6, 5, 3
6, 5, 2

6, 4, 3

5, 4, 3
5, 4, 2

4, 3, 2

There are 50 such combinations.

$$P = \frac{50}{\binom{10}{3}} = \frac{50}{120} = \frac{5}{12} .$$

15. **Method I.**

(a) This is a problem where replacements have not been made. Use combinations.

There are $\binom{5}{2}$ ways of drawing even numbers.

There are $\binom{10}{2}$ ways of drawing any kind of pair.

Thus, $$P(2 \text{ even numbers }) = \frac{\binom{5}{2}}{\binom{10}{2}} = \frac{10}{45} = \frac{2}{9} .$$

Method II.

Using $P(A \text{ and } B) = P(A)\cdot P(B)$.

$P(A) = \frac{1}{2}$; $P(B) = \frac{4}{9}$.

$P(A \text{ and } B) = \frac{1}{2}\cdot\frac{4}{9} = \frac{2}{9}$.

(b) The sum of two even numbers is an even number therefore the information in (a) is a start. The sum of two odd numbers is also an even number. Therefore we need pairs of odd numbers. There are $\binom{5}{2} = 10$ pairs of odd numbers. The probability of drawing two odd numbers is the same as drawing two evens, that is, $\frac{2}{9}$.

Since we can satisfy the requirements of the problem either by drawing two even or two odds we add the probabilities.

Thus, P(that the sum of the two numbers is even)

$$= \frac{2}{9} + \frac{2}{9} = \frac{4}{9} .$$

This naturally could have been solved by finding the number of ways of obtaining a sum which was odd.

The P(of obtaining a sum which is odd) = $\frac{25}{45}$.

(Each odd number - there are 5 - must be matched with an even number - there are 5. Thus, the numerator is $5 \times 5 = 25$).

(c) List all the ways that the sum of the two numbers is divisible by 3.

1 and 2
1 and 5
1 and 8
2 and 4
3 and 6
2 and 7
4 and 5
4 and 8
3 and 9
2 and 10
5 and 7
5 and 10
6 and 9
7 and 8
8 and 10

P(the sum is divisible by 3) = $\frac{15}{45} = \frac{1}{3}$.

(d) P(the sum is less than 20) = 1.

(e) P(the pairs have a sum more than 20) = 0 .
Note:
(d) Cannot fail and (e) cannot happen.

16. BRAINBUSTER:

(a) P(one of the pair being a dime) equals the number of pairs with a dime divided by the number of pairs with two heads. There are 3 pairs with a dime - DP, DN, and DQ. There are $\binom{4}{2} = 6$ possible pairs with two heads. $\therefore$ P(one of the pair a dime) = $\frac{3}{6} = \frac{1}{2}$.

(b) P(one of the three heads a dime) equals the number of ways a dime may be a member of the three coins with heads showing divided by the number of ways there can be three heads showing. There are $\binom{3}{2} =$ 3 ways that a dime may be one of the coins. (DPN, DPQ, DNQ) There are $\binom{4}{3} = 4$ ways that three heads may show. P(one of three heads a dime) = $\frac{3}{4}$.

17. **BRAINBUSTER:**

(a) Probability equals the number of ways that (DQ?) may occur with all heads showing divided by the number of ways that (???) may occur with all heads showing.

$$P = \frac{\binom{3}{1}}{\binom{5}{3}} = \frac{3}{10} .$$

(b) Probability equals the number of ways that (D?) may occur with heads divided by the number of ways that (??) may occur with heads showing.

$$P = \frac{\binom{4}{1}}{\binom{5}{2}} = \frac{4}{10} = \frac{2}{5}$$

(c) P(2 heads and one a dime) = P(2 heads)·P(one of the pair is a dime).

$$P(\text{2 heads}) = \frac{\binom{5}{2}}{2^5} = \frac{10}{32} = \frac{5}{16} .$$

$$P(\text{one of pair a dime}) = \frac{\binom{4}{1}}{\binom{5}{2}} = \frac{4}{10} = \frac{2}{5} .$$

$$P(\text{2 heads and one a dime}) = \frac{5}{16} \cdot \frac{2}{5} = \frac{1}{8} .$$

(d) Event A: 3 coins come up heads.

Event B: 2 of these 3 are a dime and a quarter.

$$P(\text{A and B}) = P(A) \cdot P(B).$$

$$P(A) = \frac{\binom{5}{3}}{2^5} = \frac{10}{32} = \frac{5}{16}.$$

$$P(B) = \frac{\binom{3}{1}}{\binom{5}{3}} = \frac{3}{10} .$$

$$P(\text{A and B}) = \frac{5}{16} \cdot \frac{3}{10} = \frac{3}{32} .$$

SAMPLE QUESTIONS

True-False

(T) 1. If the probability of winning the football game next Friday is $\frac{3}{5}$ the team is more likely to win than to lose.

(T) 2. If a box contains two red and two black marbles, the probability of picking a red marble is the same as the probability of picking a black marble on the first draw.

(F) 3. If a penny is tossed 20 times, we expect to have heads show exactly 10 times.

(F) 4. Probability is always a number less than $\frac{15}{16}$.

(T) 5. If 100 pennies are tossed, we can expect exactly two heads to show about 25 times.

(T) 6. The probability of "X" winning the race is $\frac{3}{7}$. "X" has less than an even chance of winning.

(T) 7. Decision making can be based on a knowledge of chance.

(T) 8. A statement such as a weather forecast is a chance statement.

(F) 9. An honest coin is one that when tossed and allowed to fall freely will always show heads.

(T) 10. Probability is sometimes used by governments.

(F) 11. If 100 pennies are tossed, we can expect that two heads will show exactly 25 times.

(F) 12. If two coins are tossed, three outcomes are possible: two heads, two tails, a head and a tail. Thus the probability of tossing two heads is one-third.

(T) 13. If five coins are tossed, the probability that two are heads is the same as the probability that three are tails.

(F) 14. Drawing a spade and drawing an ace from a deck of cards are mutually exclusive events.

(T) 15. If events A and B are independent the probability of the event: A and B, is the product of the probability of A and the probability of B.

Multiple Choice

(D) 1. Suppose you toss a coin five times and each time it shows heads. The probability that heads will show on the next toss is . . .

A. $\frac{5}{6}$

B. $\frac{1}{6}$

C. $\frac{1}{5}$

D. $\frac{1}{2}$

E. None of the above is correct.

(A) 2. Mark has three cards. One card has the letter A printed on it, another the letter C, and the third has a T. If he mixes (shuffles) the cards with the letters face down and turns them over one at a time, what is the probability that they will be turned over in the order CAT?

A. $\frac{1}{6}$

B. $\frac{1}{3}$

C. $\frac{1}{2}$

D. $\frac{3}{8}$

E. None of the above is correct.

(C) 3. A weatherman's forecasts are correct 20 times in one month and not correct 10 times in the same month. Based on this information, the estimate of the measure of chance that his future forecasts will be correct is . . .

A. $\frac{1}{3}$

B. $\frac{1}{2}$

C. $\frac{2}{3}$

D. $\frac{3}{4}$

E. None of the above is correct.

(B) 4. A drawer contains 10 blue stockings and 10 red stockings. What is the smallest number of stockings you need to take to be sure you have a pair of the same color stockings?

A. 2

B. 3

C. 5

D. 10

E. 11

(A) 5. There are two roads from Milltown to Luck and three roads from Luck to Turtle Lake. How many different ways can John travel from Milltown to Turtle Lake?

A. 6

B. 5

C. 3

D. 2

E. None of these.

(E) 6. If six honest coins are tossed and allowed to fall freely, how many possible outcomes are there?

A. 4
B. 8
C. 16
D. 32
E. None of the above is correct.

(D) 7. Each face of two regular tetrahedrons is painted a particular color, red, blue, yellow, and green. If the tetrahedrons are rolled what is the chance that both solids will stop on the white faces?

A. $\frac{1}{2}$
B. $\frac{1}{4}$
C. $\frac{1}{8}$
D. $\frac{1}{16}$
E. None of the above is correct.

Computation Questions

Compute the measures of chance for these events.

($\frac{1}{3}$) 1. A box contains a red marble, green marble and yellow marble. If you draw two marbles without looking, what are the chances that you draw the red and green marble in any order.

($\frac{2}{3}$) 2. Three jackets are in a dark closet. One jacket belongs to you, one to Bill, and one to Bob. If you reach into the closet and take out two jackets, what is the chance that you get your own jacket?

(8) 3. How many equally likely events are possible if a regular octahedron (an eight-sided solid) is rolled?

(64) 4. How many outcomes are possible if two regular octahedrons are rolled simultaneously (at the same time)?

($\frac{2}{3}$) 5. Three cards are numbered like the ones below.

$\boxed{3}\quad\boxed{5}\quad\boxed{6}$

Without looking at the numbers you are to draw any two cards. What is the chance that the sum of the two numbers on the cards you draw is odd?

6. Assume that the probability of a family having a boy is $\frac{1}{2}$, and having a girl is also $\frac{1}{2}$.

($\frac{1}{8}$) (a) What is the probability of a family having three boys in succession?

($\frac{7}{8}$) (b) What is the probability in a family with three children of not having three girls?

($\frac{1}{8}$) (c) What is the probability of a family having two boys and a girl, in that order?

($\frac{3}{8}$) (d) What is the probability of a family with three children having two boys and a girl, in any order?

($\frac{3}{8}$) (e) What is the probability of a family with four children having two boys and two girls, in any order?

($\frac{1}{16}$) (f) What is the probability of a family with four children having four girls?

($\frac{5}{8}$) (g) What is the probability in a family of four children of not having half boys, half girls?

(one out of 4096) (h) Occasionally one reads in the newspaper of a family that has twelve children, all of them boys. In families that have twelve children, how often is this likely to happen?

Chapter 9

SIMILAR TRIANGLES AND VARIATION

Introduction

This chapter involves ideas from both algebra and geometry. The topics to be studied by sections are:

9-1. Introduction to similar right triangles through scale drawing.

9-2. Definition of similar triangles and illustration of properties of similar right triangles using congruent right triangles.

9-3. Definition of the trigonometric ratios of sine, cosine, and tangent; construction of tables using measurement; trigonometric ratios of 30°, 45°, and 60° angles.

9-4. Equations of lines in the form $y = ax$ and slope of a line.

9-5. Reading a table of trigonometric ratios.

9-6. Kinds of variation.

9-7. Direct variation.

9-8. Inverse variation.

9-9. Other types of variation (optional).

Recommended teaching time: 20 days; 22 days if Section 9-9 is included.

A summary, and review exercises, are given in 9-10.

The chapter involves two basic ideas: similar right triangles and variation. In the 1959-60 or first edition of the course for Grade 8 these ideas were developed in separate chapters. In this less formal treatment of similar right triangles it seemed possible to combine the materials into a single chapter. Much of the treatment of similar right triangles is based on a consideration of the graph of $y = ax$ which is the number sentence for the relation of direct variation.

9-1 Indirect Measurement

Several examples are given of distances and lengths which obviously would have to be measured indirectly. Then finding the height of a building, using the angle of elevation measured by a sextant, is introduced. The teacher may wish to discuss use of a sextant or other simple instruments by which an angle of elevation may be measured. In some classes a review of scale drawings in Chapter 4 would be appropriate. That a scale drawing of a physical model is the same "shape" as the model is to be emphasized.

In the exercises the students are asked to measure the sides of a number of right triangles which contain an angle of 60° and to determine ratios of pairs of corresponding sides. The approximate nature of the ratios determined in this way should be carefully observed. Use of graph paper for the drawings simplifies the problem of measuring. If these exercises are done as class exercises a comparison of measures and ratios can be made profitably and the work on the part of any one student would be less burdensome.

Exercises 9-1 which might also be done as class exercises are intended to show that the ratios of corresponding pairs of sides of two right triangles are equal if the corresponding angles of the two triangles are congruent. Use is made of congruent triangle properties which will provide desirable review and maintenance of Chapter 5. The arguments which students are to provide are, of course, proofs only in special cases. However, the idea of similar triangles and their relation to congruent triangles should be more clear for students when these exercises have been completed.

Answers to Class Discussion Exercises in Section 9-1

1. CB ≈ 3.46 in. (3.46)(50)ft. ≈ 173 ft.

2-4. They have the same shape.

Right triangle					
ABC	$y_1 = CB = \frac{\sqrt{3}}{2}$	$x_1 = AC = \frac{1}{2}$	$\frac{y_1}{x_1} = \frac{CB}{AC} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$	$\frac{x_1}{r_1} = \frac{AC}{AB} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$	$\frac{y_1}{r_1} = \frac{CB}{AB} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2}$
ADE	$y_2 = ED = \sqrt{3}$	$x_2 = AE = 1$	$\frac{y_2}{x_2} = \frac{ED}{AE} = \frac{\sqrt{3}}{1} = \sqrt{3}$	$\frac{x_2}{r_2} = \frac{AE}{AD} = \frac{1}{2}$	$\frac{y_2}{r_2} = \frac{ED}{AD} = \frac{\sqrt{3}}{2}$
AFG	$y_3 = GF = \frac{3\sqrt{3}}{2}$	$x_3 = AG = 1\frac{1}{2}$	$\frac{y_3}{x_3} = \frac{GF}{AG} = \frac{\frac{3}{2}\sqrt{3}}{\frac{3}{2}} = \sqrt{3}$	$\frac{x_3}{r_3} = \frac{AG}{AF} = \frac{\frac{3}{2}}{3} = \frac{1}{2}$	$\frac{y_3}{r_3} = \frac{GF}{AF} = \frac{\frac{3\sqrt{3}}{2}}{3} = \frac{\sqrt{3}}{2}$
AHI	$y_4 = IH = 2\sqrt{3}$	$x_4 = AI = 2$	$\frac{y_4}{x_4} = \frac{IH}{AI} = \frac{2\sqrt{3}}{2} = \sqrt{3}$	$\frac{x_4}{r_4} = \frac{AI}{AH} = \frac{2}{4} = \frac{1}{2}$	$\frac{y_4}{r_4} = \frac{IH}{AH} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

3. See the next to the last column in the above table.

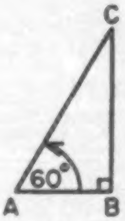
4. See the last column in the table above.

5-6. See Problems 6 and 7 of Exercises 9-4.

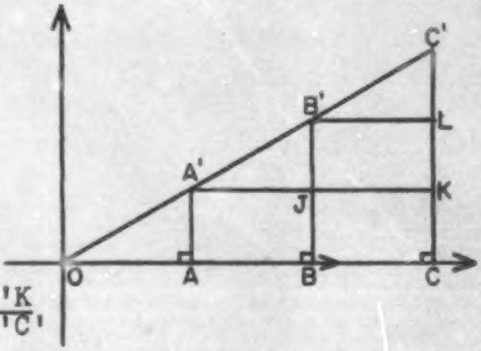
Answer to discussion Question prior to Exercises 9-1.

$\frac{AB}{BC} = \sqrt{3}$; $AB = BC\sqrt{3}$ and

$BC = \frac{AB}{\sqrt{3}}$ or $\frac{AB\sqrt{3}}{3}$



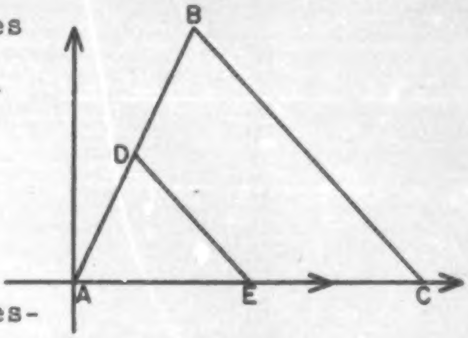
Answers to Exercises 9-1



1. (a) $\angle A'OA \cong \angle B'A'J \cong \angle C'B'L$
   (b) $\angle OA'A \cong \angle A'B'J \cong \angle B'C'L$
   (c) 90
   (d) $\Delta A'JB' \cong \Delta OAA' \cong \Delta B'LC'$
   (e) $\Delta OBB' \cong \Delta A'KC'$
   (f) $\frac{OA}{OA'} = \frac{A'J}{A'B'} = \frac{B'L}{B'C'} = \frac{OB}{OB} = \frac{A'K}{A'C'}$
   (g) $\frac{OA}{AA'} = \frac{A'J}{B'J} = \frac{B'L}{C'L} = \frac{OB}{B'B} = \frac{A'K}{C'K}$
   (h) $\frac{B'J}{A'J} = \frac{A'A}{OA} = \frac{C'L}{B'L} = \frac{B'B}{OB} = \frac{C'K}{A'K}$.

2. No.

3. Corresponding sides are congruent, corresponding angles are $\cong$; therefore the measures of the corresponding sides are equal and the measures of the corresponding $\angle$ s are equal.



4. (a) $\frac{AB}{AC} = \frac{AD}{AE}$ Corresponding sides of $\sim$ $\triangle$s are proportional.
   (b) $\frac{AB}{AD} = \frac{AC}{AE}$ Same reason.
   (c) $\frac{AD}{DE} = \frac{AB}{BC}$ Same reason.
   (d) $\frac{AD}{AC} \neq \frac{AB}{AE}$ They are not corresponding sides of similar Δ s and are therefore not proportional.

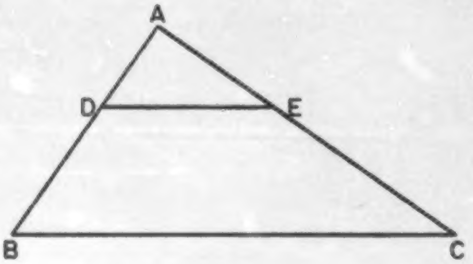
5. No, the same conditions would hold, namely corresponding sides of similar triangles are proportional.

6. (a) $\frac{AB}{AC} = \frac{AD}{AE}$
(b) $\frac{AB}{AD} = \frac{AC}{AE}$
(c) $\frac{AD}{DE} = \frac{AB}{BC}$

If two Δs are $\sim$, the ratio of pairs of corresponding sides are equal.

(d) $\frac{AD}{AC} \neq \frac{AB}{AE}$

They are not pairs of corresponding sides of $\sim$ Δs, therefore they do not form equal ratios.



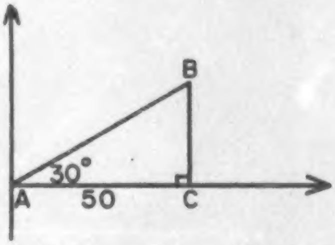
7. 90-31 = 59

8. 180-(35 + 47) = 180-(82) = 98

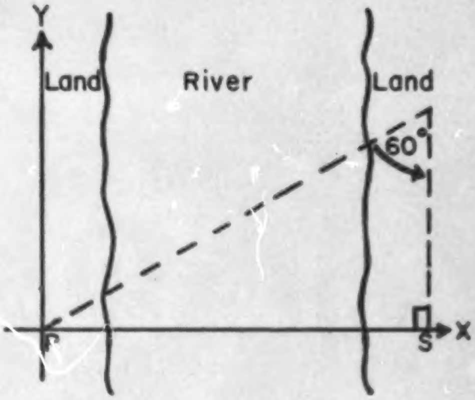
9. $\angle C = \angle C'$
$\angle A = \angle A'$
$(\angle A + \angle C) = (\angle A' + \angle C')$
$\angle A + \angle B + \angle C = \angle A' + \angle B' + \angle C' = 180$, $\therefore \angle B = \angle B'$

10. Same as above.

11. $\frac{BC}{AC} = \frac{\sqrt{3}}{3}$
$BC = (AC)\frac{\sqrt{3}}{3}$
$BC = 50\frac{\sqrt{3}}{3}$
$BC \approx 29$
Students should not be expected to give this solution. From a scale drawing the answer should be 28 or 29.



12. By scale drawing the answer obtained will be about 1700. Using tan 60° = 3 one obtains 1732. The drawing here is not in the scale requested in the problem.



9-2 Similar Triangles

While our concern in this chapter will be with similar right triangles there is no new difficulty involved in giving a rather precise definition of similar triangles, without the restriction to right triangles. It is easy for teacher and student (and text author!) to use the terms "corresponding angles" and "ratios of corresponding pairs of sides" rather loosely. Occasional use and reference to the precise definition given in this section is quite desirable. Students may find the topic of similar triangles quite difficult just because the words and terms we often use are used too carelessly.

Again, of course, there is no intent to provide proofs, or even informal arguments, for the properties stated in this section. The experience provided in Section 9-1 should make the conclusions appear reasonable. That is the best we can hope for in this kind of treatment. The logical points involved in proving these properties are among the most difficult and subtle of all elementary geometry and are left for the formal geometry course, usually studied in Grade 10.

Answer to Exercises 9-2

1. (a) $m(\angle A') = 30$, $m(\angle B') = 75$

   (b) $A'C' = 8$

   (c) $\frac{2}{3} = \frac{5}{B'C'}$, $B'C' = \frac{3}{2}(5) = 7\frac{1}{2}$. $A'C'$ cannot be determined since there is insufficient information given.

   (d) $A'B'$ cannot be determined - insufficient information.

   $\frac{4}{5} = \frac{B'C'}{3}$; $B'C' = \frac{4}{5}(3) = 2\frac{2}{5}$.

   (e) $m(\angle A') = 30$, $m(\angle C')$ cannot be determined - insufficient information.

2.

(T) (a) Alternate definition 1.

(F) (b) The third pair of corresponding sides are not necessarily congruent.

(T) (c) Alternate definition 2.

(F) (d) Insufficient information to make the claim that $\triangle ABC \sim \triangle A'B'C$.

(T) (e) Alternate definition 2.

*3. (a) No; for example take a square and a rectangle which is not a square; the corresponding angles are equal, but the corresponding sides are not proportional.

   (b) No; consider two parallelograms whose sides are proportional; their corresponding angles need not be congruent.

9-3 The Trigonometric Ratios

Since the trigonometric ratios are so useful in solving problems of indirect measurement it seems important to introduce them in this chapter. These topics are now to be found in many widely used textbooks for Grades 8, 9, and 10. However, the SMSG texts for Grades 9 and 10 do not include these ratios and hence it becomes all the more advisable that they have a place in the Grade 8 course. In this introduction the ratios for sine, cosine, and tangent are discussed, tables for them are to be constructed using measurement, and applications are given.

The property of the equality-of-ratios of corresponding sides of similar right triangles is re-emphasized in the development leading to construction of the tables. It is first suggested that measurements might be made for many 1° right triangles, and then the conclusion is drawn that this is not necessary because of the equality-of-ratios property.

In the actual construction of tables by students it is proposed that a circle of unit radius be used. If 10 centimeters is chosen for the unit, it is much easier to obtain measures expressed as decimals. Also, as is suggested in the exercises, if the radius of the circle is used as the hypotenuse in finding sines and cosines, and as the side adjacent to the angle in finding tangents, it is not necessary to divide in finding the ratios of numbers obtained by measurement. You will observe, of course, that in this procedure, the denominator of the ratios is 1 in all three cases. Sometimes the segments measured are called line values of the trigonometric ratios. You may wish to ask the students why this is an appropriate name.

Most students will profit from the opportunity provided to apply the Pythagorean property as suggested in the sample problems. Such application will also further emphasize for the students the approximate nature of measurements and of the trigonometric ratios.

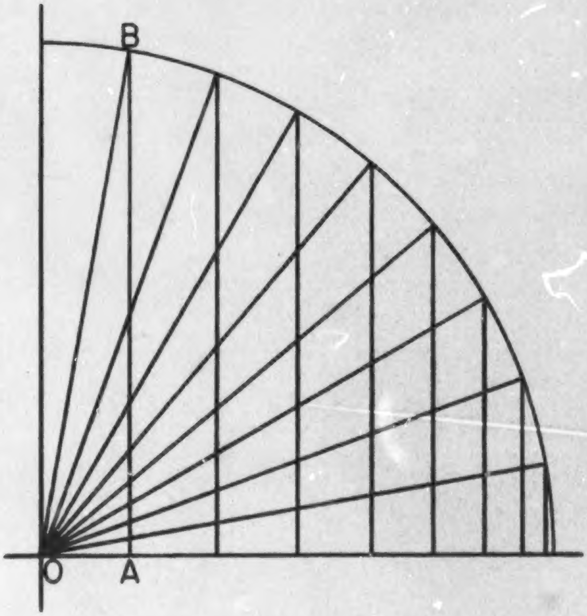
Unless the teacher feels that the students will object to computing the ratios when they are given in a table, the attention of the students might be called to the table in Section 5. Perhaps the most appropriate time to do this would be after a table has been constructed by class teamwork. A check of the classwork by the table then could be made.

The study of trigonometric ratios of special angles is not essential in the development of this chapter. The topic is introduced to show relationships of the present work to properties of triangles studied earlier. It is of course interesting to know that the ratios for angles in two special triangles can be determined from triangle properties without any measurement. After Chapter 6 on Real Numbers it is good for students to use irrational numbers in new kinds of problems.

Answers to Exercises 9-3a

1.

∠AOB	Sin ∠AOB	Cos ∠AOB
10°	0.17	0.98
20°	0.34	0.94
30°	0.50	0.87
40°	0.64	0.77
50°	0.77	0.64
60°	0.87	0.50
70°	0.94	0.34
80°	0.98	0.17



2.

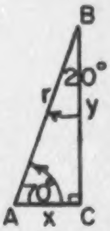
$\angle$ AOB	Tan $\angle$ AOB
10°	0.18
20°	0.36
30°	0.58
40°	0.84
50°	1.19
60°	1.73
70°	2.75
80°	5.67

3. It is not easy to obtain the necessary measures to calcu- the sine and tangent ratios for angle θ if $0^\circ < \theta < 10^\circ$; furthermore we have not defined an angle whose measure is 0. Similarly it is not easy to obtain the necessary measures to calculate the cosine and tangent ratios for angle θ if $80^\circ < \theta < 90^\circ$. Furthermore we have not defined the cos 90°. As $\theta \longrightarrow 90^\circ$ tangent θ becomes larger larger.

4.

$\measuredangle$AOB	OA	AB	OB	$\measuredangle$ABO
30°	7	$7 \text{ Sin } 30^\circ = 7(\frac{1}{2}) = 3\frac{1}{2}$	$7 \text{ Cos } 30^\circ = 7(\frac{\sqrt{3}}{2}) = 6.1$	60°
25°	$6 \text{ Sin } 65^\circ = 6(.91) = 5.5$	6	$6 \text{ Cos } 65^\circ = 6(.42) = 2.5$	65°

5. (a) $\sin 20^\circ = \cos 70^\circ$



$\sin 20^\circ = \frac{x}{r}$, $\cos 70^\circ = \frac{x}{r}$;

hence $\sin 20^\circ = \cos 70^\circ$.

6. sin $\angle$ AOB increases as $\angle$ AOB increases in measure.
cos $\angle$ AOB decreases as $\angle$ AOB increases in measure.
tan $\angle$ AOB increases as $\angle$ AOB increases in measure.

*7. $[\sin 20^\circ]^2 + [\cos 20^\circ]^2 = (0.34)^2 + (0.94)^2 \approx 1.$

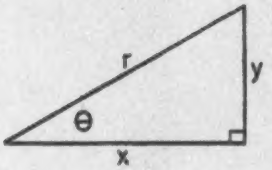
$[\sin 30^\circ]^2 + [\cos 30^\circ]^2 = (\frac{1}{2})^2 + (0.87)^2 \approx 1.$

$[\sin 80^\circ]^2 + [\cos 80^\circ]^2 = (0.98)^2 + (0.17)^2 \approx 1.$

Proof

(1) $y^2 + x^2 = r^2$

(2) $\sin \theta = \frac{y}{r}$ $\cos \theta = \frac{x}{r}$



(3) $(\sin \theta)^2 = \frac{y^2}{r^2}$ $(\cos \theta)^2 = \frac{x^2}{r^2}$

(4) $(\sin \theta)^2 + (\cos \theta)^2 = \frac{y^2}{r^2} + \frac{x^2}{r^2} =$

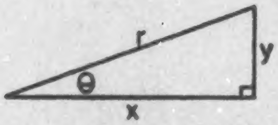
(5) $\frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1.$

*8. $\frac{\sin 20^\circ}{\cos 20^\circ} = \tan 20^\circ$

Proof

$\sin \theta = \frac{y}{r}$ $\cos \theta = \frac{x}{r}$ $\tan \theta = \frac{y}{x}$

$\frac{\sin \theta}{\cos \theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x} = \tan \theta$

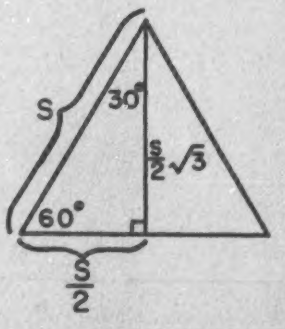


Answers to Exercises 9-3b

1. $\sin 30^\circ = \frac{\frac{s}{2}}{s} = \frac{1}{2}$

$\cos 30^\circ = \frac{\frac{s}{2}\sqrt{3}}{s} = \frac{\sqrt{3}}{2}$

$\tan 30^\circ = \frac{\frac{s}{2}}{\frac{s\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$



2. (a) $m(\angle ABO) = 45$.

(b) OAB is a right triangle. It is also isosceles.

(c) The length of $\overline{AB}$ is 1 unit.

(d) The length of $\overline{OB}$ is $\sqrt{2}$ units.

(e) $\tan 45^\circ = \tan \angle AOB = \frac{AB}{OA} = \frac{1}{1} = 1.$

$\sin 45^\circ = \sin \angle AOB = \frac{AB}{OB} = \frac{1}{\sqrt{2}}.$

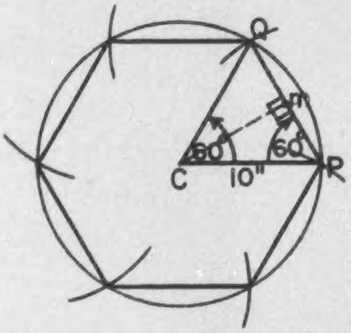
$\cos 45^\circ = \cos \angle AOB = \frac{OA}{OB} = \frac{1}{\sqrt{2}}.$

Using $\sqrt{2} \approx 1.4142$ we obtain $\tan 45^\circ = 1.0000$,
$\sin 45^\circ = \cos 45^\circ \approx \frac{1}{1.4142} = .7071$

3. $\sin 45^\circ = \cos 45^\circ = \frac{1}{2}\sqrt{2} \approx \frac{1}{2}(1.4142) = .7071.$

4. $\frac{\sin 60^\circ}{\sin 30^\circ} = \sqrt{3}$ $\qquad \frac{\tan 60^\circ}{\tan 30^\circ} = \frac{\sqrt{3}}{\frac{1}{\sqrt{3}}} = \sqrt{3}\cdot\sqrt{3} = 3.$

$\sin 60^\circ$ is not $2 \sin 30^\circ$
$\tan 60^\circ$ is not $2 \tan 30^\circ$.



5. (a) $m(\angle PCQ) = 60$

(b) $m(\angle CPQ) = 60$

(c) $CM = \frac{10\sqrt{3}}{2} \approx \frac{10(1.732)}{2} \approx 8.66$

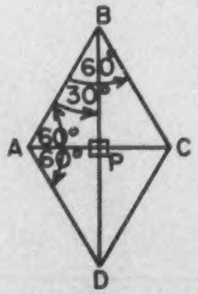
(d) 10

*6. (a) $m(\angle ABD) = 30$

(b) $m(\angle DBC) = 30$

(c) $\overline{BD} \perp \overline{AC}$ because $m(\angle APB) = 90$

(d) $BD = 2BP$



9-4 Slope of $y = ax$

Study of the slope of lines has usually been reserved for more advanced courses in mathematics. The slope property follows naturally from our study this far and serves as the first example of the constant of proportionality to be studied later. For most students it is quite an easy topic. The development of the equations of the line, $y = ax$, may prove to be more difficult. It will be helpful to students in later study of mathematics and science if they can at this time understand that there are two aspects to the property, the graph of a line has a certain equation and that an equation has a certain graph.

1. If (r,s) is a point on the graph of $y = ax$, then $s = ar$.
2. If $s = ar$, then the point (r, s) is a point on the graph of $y = ax$.

The first of these statements is proved in the text and the second is left for the student as a problem.

If a line makes an angle of measure θ with X-axis, then $\tan \theta$ is the slope of the line. In this treatment only lines are considered where $0<\theta<90$. For these lines the slope is positive. In better classes the teacher may wish to develop the idea that the slope of the line determined by the points $(1, 4)$ and $(6, 7)$ is $\frac{7-4}{6-1} = \frac{3}{5}$. This is an application of the definition of slope as "the change in y divided by the change in x." More precisely we can say, if the points (a, b) and (c, d) lie on line ℓ, the slope of ℓ is $\frac{d-b}{c-a}$.

The exercises on finding products of real numbers, using the number line and similar right triangles associated with the number line, can be a very important part of the chapter. Now that students have learned more about the graph of the equation, $y = ax$, and have some ability to see the relations involved in similar right triangles, they are in a good position to extend their understanding of real numbers. Thus this material adds to the ideas studied in Chapter 6, provides for review of part of Chapter 6, and gives a very interesting and different kind of application of the basic ideas of this chapter.

Answers to Questions in Section 9-4a

The line $\overleftrightarrow{OP}$ intersects the three vertical lines in the points (3, 6), (4, 8), and (2, 4). In each case $\frac{y}{x} = 2$. For each of these points $y = 2x$. Triangles OAP and OBQ are right triangles. They have a pair of acute angles which are equal in measure. In similar triangles the ratios of the measures of the lengths of corresponding sides are equal. We conclude that $\frac{BQ}{OB} = \frac{AP}{OA}$, and that $\frac{y}{x} = 2$ for any point on $\overleftrightarrow{OP}$.

$$\text{If } \frac{y}{x} = 2 \text{ then } y = 2x.$$

Each of the points $(^{-}1, ^{-}2)$, $(^{-}2, ^{-}4)$, and $(^{-}3, ^{-}6)$ lie on $\overleftrightarrow{OP}$, and for each of these points $\frac{y}{x} = 2$. If x and y are the coordinates of any point on $\overleftrightarrow{OP}$, then

$$y = 2x .$$

The points (4, 6), (6, 9), and $(1, \frac{3}{2})$ all lie on $\overleftrightarrow{OP'}$. For each the ratio $\frac{y}{x}$ is $\frac{3}{2}$.

Triangles AOP' and B'OQ' are similar. We know that $\frac{y}{x} = \frac{3}{2}$ since ratios of the measures of corresponding sides of similar triangles are equal. This relation can be rewritten $y = \frac{3}{2}x$.

Answers to Exercises 9-4a

1. $y = \frac{1}{4}x$

   $y = \frac{1}{3}x$

   $y = x$

   $y = 2x$

   $y = 5x$

2. Tan $\angle AOP' = \frac{3}{2} = 1.5000$.

3. $y = \frac{3}{2}x$

   $y = \frac{3}{2}(9)$

   $y = 13.5$

   $\frac{\text{change in } y \text{ from } Q \text{ to } R}{\text{change in } x \text{ from } Q \text{ to } R} = \frac{7.5}{5} = \frac{3}{2}$

4. (a) See text

   (b) See text

   (c) $RK = \frac{7}{3}OR = RK = \frac{7}{3}$.

   coordinates of K are $(1, \frac{7}{3})$

   (d) See text

   (e) $SL = 2(\frac{7}{3})$ or $\frac{14}{3}$.

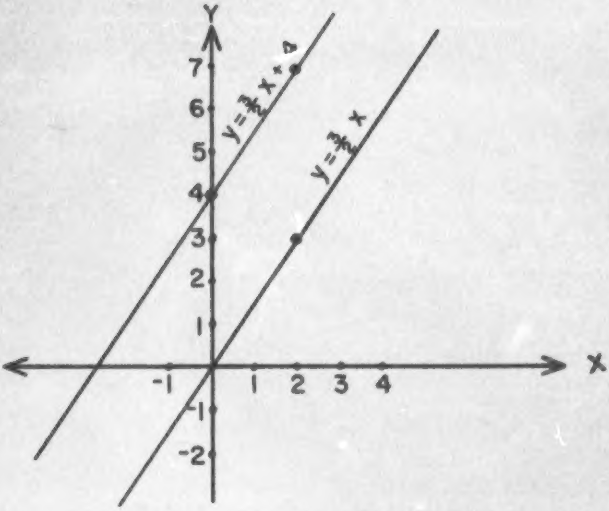
   (f) $\frac{RK}{OR} = \frac{\frac{7}{3}}{1} = \frac{7}{3}, \quad \frac{SL}{OS} = \frac{\frac{14}{3}}{2} = \frac{7}{3}$.

5. (a) $d' = mc$ is true since Q' is on $\overleftrightarrow{OP}$, and the coordinates of any point on $\overleftrightarrow{OP}$ satisfy the equation $y = mx$.

   (b) $d = d'$ (because $d = mc = d'$). Q and Q' coincide.

   (c) If the coordinates of Q satisfy the equation $\underline{y = mx}$ then Q lies on the line $\underline{\overleftrightarrow{OP}}$.

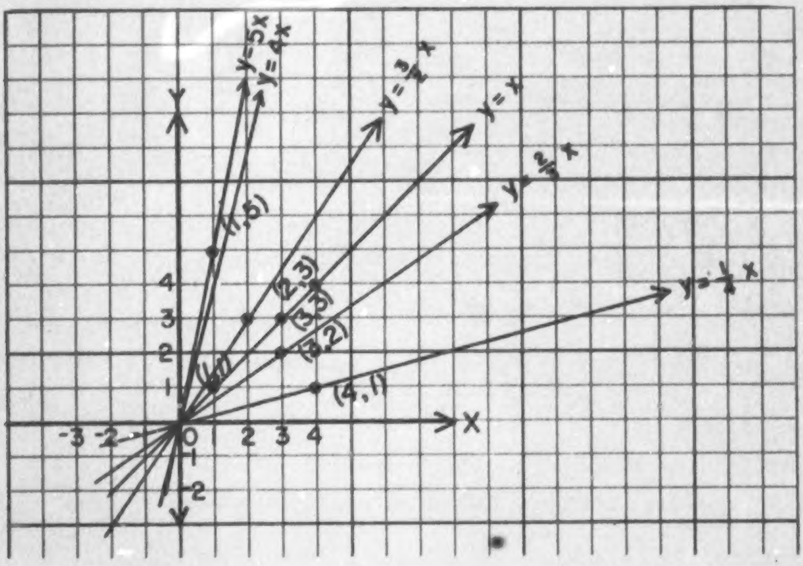
6. BRAINBUSTER:



The two lines do not intersect. The tangent of the angle which this line makes with the X-axis is $\frac{3}{2}$.

Answer to Class Questions in Section 9-4b

For Problem 1, $m = \frac{1}{4}, \frac{1}{3}, 1, 2, 5$.



$m = \frac{1}{4}$ is the smallest

$m = 5$ is the largest.

As m increases the line becomes steeper.

Answer to Exercises 9-4b

1.

(x, y) for $y = \frac{2}{3}x$	(x,y) for $y = \frac{1}{4}x$	(x,y) for $y = 4x$
(0,0)	(0,0)	(0,0)
(3,2)	(4,1)	(1,4)
(6,4)	(8,2)	(2,8)
(9,6)	(12,3)	(3,12)

See graphs above for graphs of $y = \frac{2}{3}x$, $y = \frac{1}{4}x$, $y = 4x$

for $y = \frac{2}{3}x$ the slope is $\frac{2}{3}$

for $y = \frac{1}{4}x$ the slope is $\frac{1}{4}$

for $y = 4x$ the slope is 4

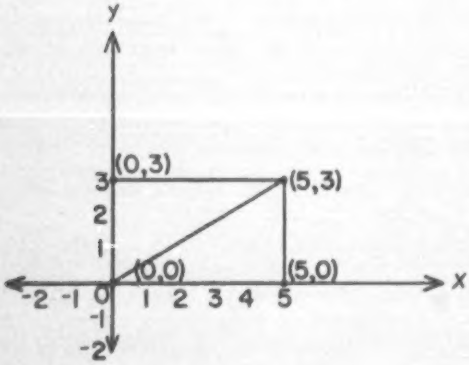
2. (a) The slope = 3

(b) The slope = $\frac{3}{2}$

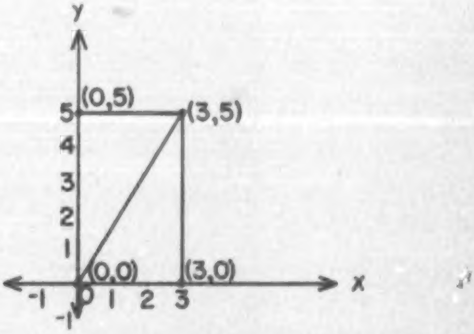
(c) The slope = 10

(d) The slope = 4

3.


y
(0,3)
(5,3)
3
2
1
(0,0)
(5,0)
x
-2 -1 0 1 2 3 4 5
-1
-2

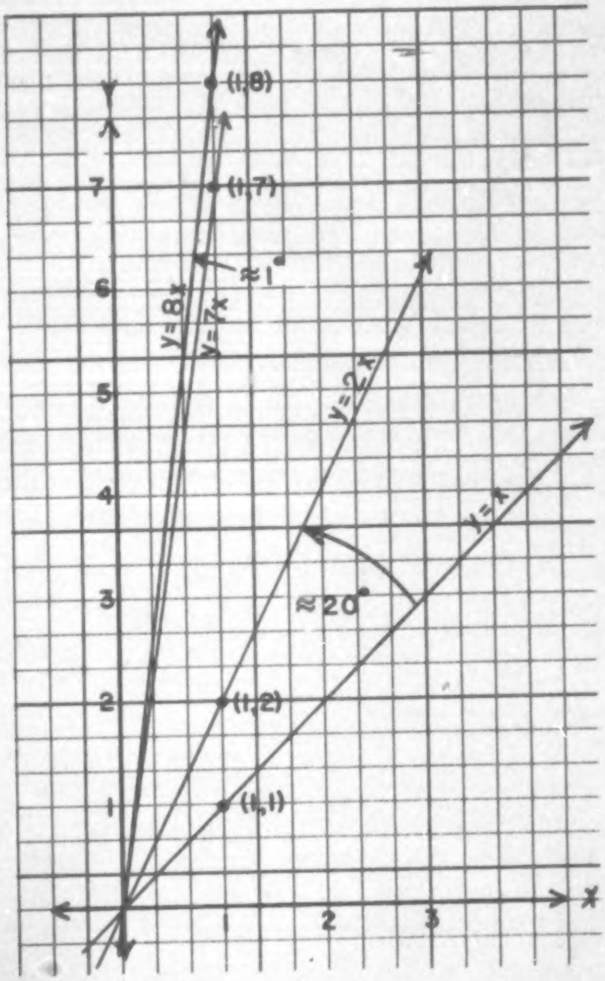
The slope of the diagonal is $\frac{3}{5}$.


y
(0,5)
(3,5)
5
4
3
2
1
(0,0)
(3,0)
x
-1 0 1 2 3
-1

The slope of the diagonal is $\frac{5}{3}$.

4. $\frac{\text{vertical rise}}{\text{horizontal run}} = \frac{10}{5280} = \frac{1}{528}$

5. The measure of the angle formed by the ray from (0,0) on the lines $y = x$ and $y = 2x$ is approximately 20 times the measure of the angle formed by the rays from (0,0) on the lines $y = 7x$ and $y = 8x$.



Answers to Exercise 9-4c

1. (a) See diagram

   (b) See diagram, $y = \frac{3}{2}x$

   (c) $(2, 3)$

   $(3, \frac{9}{2})$

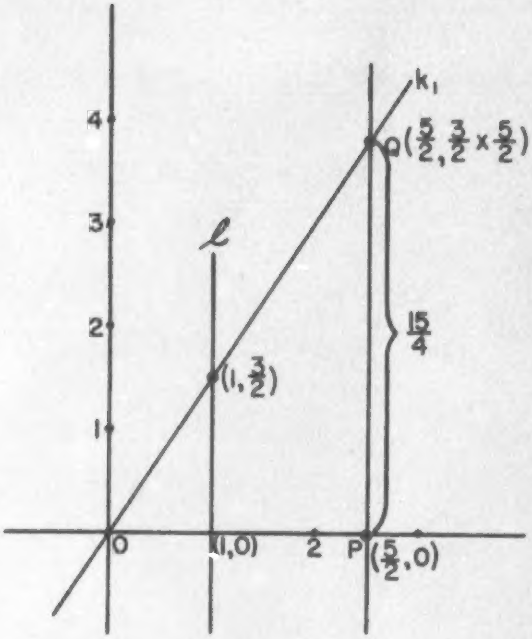
   $(^-3, ^-\frac{9}{2})$

   $(\frac{4}{3}, 2)$

   $(\frac{22}{7}, \frac{33}{7})$

   $(b_1, \frac{3}{2}b_1)$ no you cannot.

   (d) $\frac{3}{2} \times \frac{5}{2} = \frac{15}{4} = 3\frac{3}{4}$


k_1
4
3
2
1
ℓ
$Q(\frac{5}{2}, \frac{3}{2} \times \frac{5}{2})$
$(1, \frac{3}{2})$
$\frac{15}{4}$
O
(1,0)
2
P
$(\frac{5}{2}, 0)$

2. Construction A = $\sqrt{2}$

   B = $\sqrt{3}$

   C = $\sqrt{4} = 2$

   D = $\sqrt{5}$

   E = $\sqrt{6}$

3. (a) See construction

(b) Coordinates of A are $(1, \sqrt{2})$ slope of $k_2 = \sqrt{2}$

Equation of k_2

$y = (\sqrt{2})(x)$

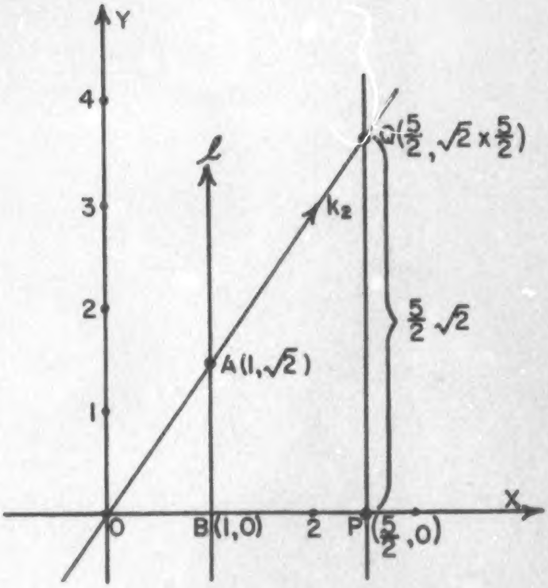
(c) $(2, 2\sqrt{2})$

$({}^-1, {}^-\sqrt{2})$

$({}^-2, {}^-2\sqrt{2})$

$(b, b\sqrt{2})$

(d) $PQ = \frac{5}{2}\sqrt{2}$


Y
4
3
2
1
ℓ
k_2
$Q(\frac{5}{2}, \sqrt{2} \times \frac{5}{2})$
$\frac{5}{2}\sqrt{2}$
$A(1, \sqrt{2})$
X
O
B(1,0)
2
$P(\frac{5}{2}, 0)$

4. (a) Construction

(b) Slope of $k_3 = a$

$y = ax$

(c) $(2, 2a)$

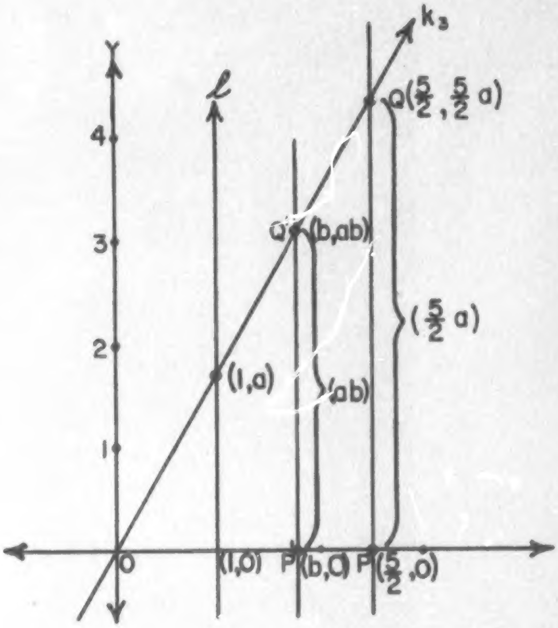
$({}^-1, {}^-a)$

$({}^-2, {}^-2a)$

(b, ab)

(d) $PQ = \frac{5}{2}a$

(e) $P'Q' = ab$


k_3
Y
ℓ
$Q(\frac{5}{2}, \frac{5}{2}a)$
4
3
2
1
$Q'(b, ab)$
$(\frac{5}{2}a)$
$(1, a)$
(ab)
O
(1,0)
$P'(b,0)$
$P(\frac{5}{2}, 0)$

5. (a) Set up the coordinate axes.

(b) Construct the line $\ell \perp$ to the X-axis through the point (1,0).

(c) Mark off the number a on ℓ, with one end point of segment a at (1,0). Label the coordinates of the other end point of segment a on ℓ.

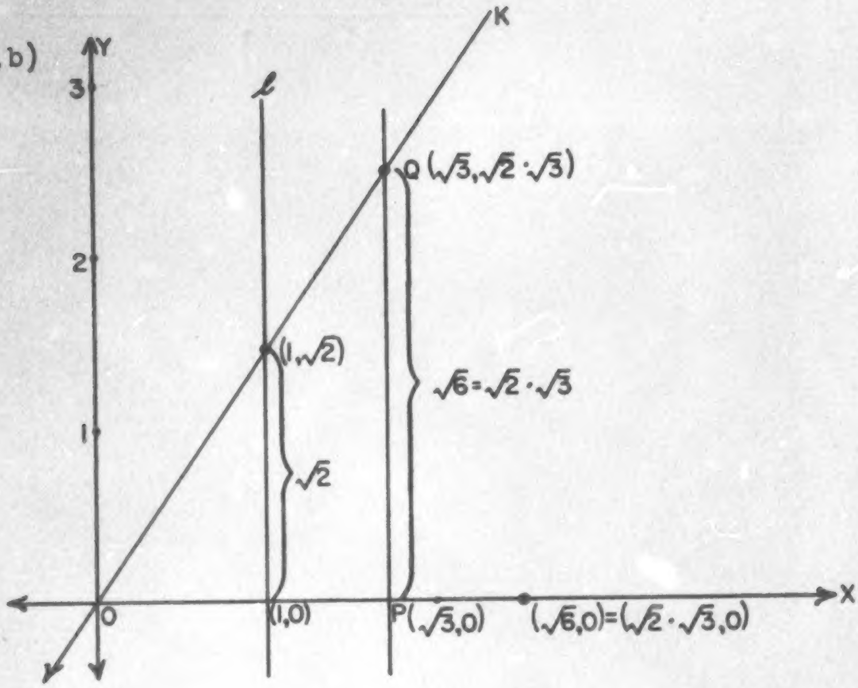
(d) Join the origin to the end point of a on ℓ whose coordinates are (1,a). Call the line containing these points line k. Its equation is $y = ax$.

(e) Lay off the segment b on the X-axis with one end point at (0,0). The coordinates of the other end point of segment b on the X-axis are P(b,0).

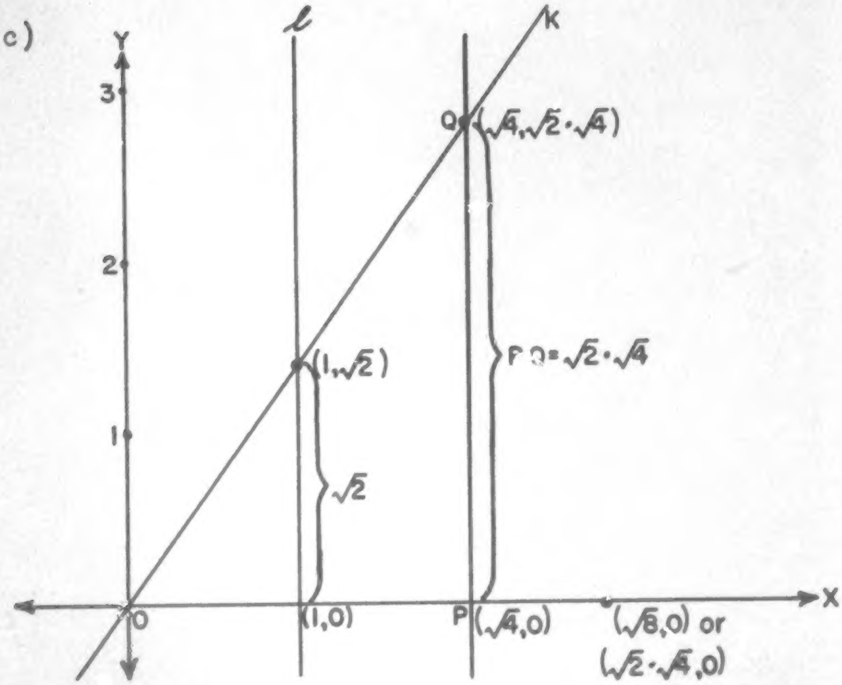
(f) Construct a $\perp$ to the X-axis at (b,0). Extend the $\perp$ until it intersects k at point Q. The coordinates of Q are (a,ab). Thus PQ = ab.

(g) Lay off ab on the X-axis with one end point at the origin. This measure of line segment ab on the X-axis is the product of the measures of line segment a and line segment b.

6. (a,b)


Y
3
2
1
ℓ
K
Q(√3, √2·√3)
(1, √2)
√6 = √2·√3
√2
O
(1,0)
P(√3,0)
(√6,0) = (√2·√3,0)
X

(c)


Y
3
2
1
ℓ
K
Q(√4, √2·√4)
(1, √2)
PQ = √2·√4
√2
O
(1,0)
P(√4,0)
(√8,0) or
(√2·√4,0)
X

(d) $\sqrt{3}\cdot\sqrt{5} = \sqrt{15}$

$\sqrt{5}\cdot\sqrt{11} = \sqrt{55}$

$\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ if $a > 0$, $b > 0$.

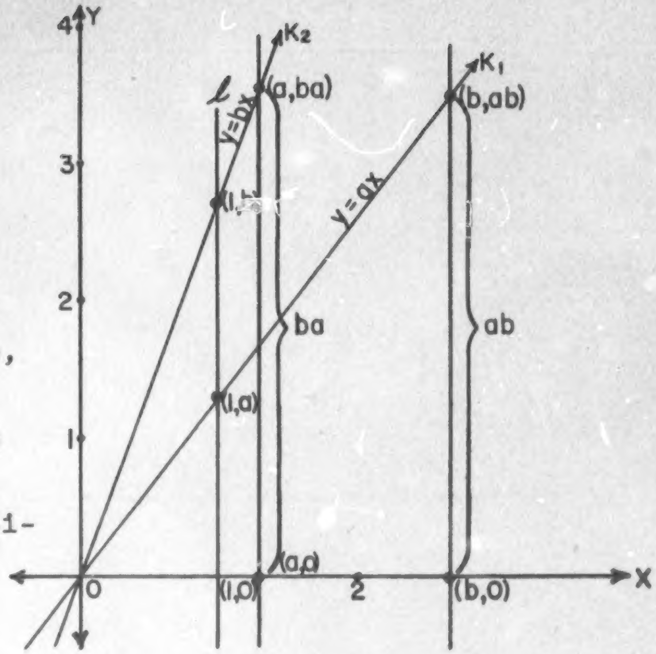
7. (a) Construction

$k_1 : y = ax$

$k_2 : y = bx$

The lengths are equal.

∴ the measures ab, ba are equal. This illustrates the commutative property of multiplication for real numbers.


Y
4
3
2
1
0
2
X
K₂
K₁
ℓ
y=bx
y=ax
(a,ba)
(b,ab)
(1,b)
(1,a)
(a,0)
(1,0)
(b,0)
ba
ab

(b) $r = ca$

$s = ca$

$t = w + s$ or $c(a + b)$

$u = b$

$v = b$

$\frac{w}{r} = \frac{v}{a}$

$w = \frac{rv}{a}$

$w = \frac{acb}{a}$

$w = cb$

$c(a + b) = t$

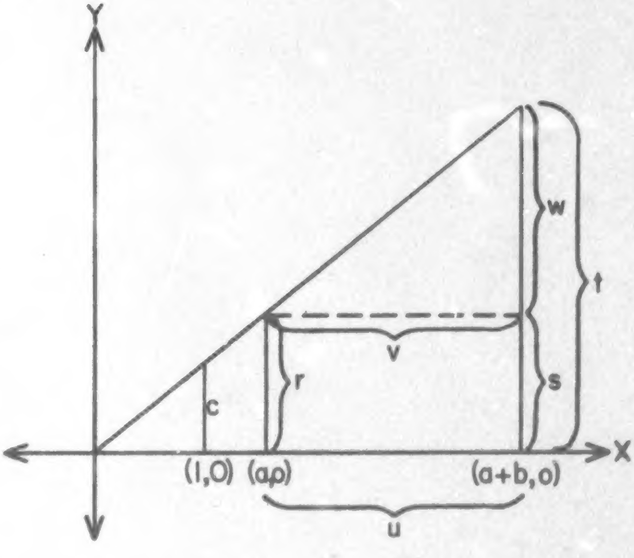
$w + s = t$

$cb + ca = t$

$ca + cb = t$

thus $c(a + b) = ca + cb$

This is the distributive property of multiplication over addition for the real numbers.


Y
X
w
t
v
r
s
c
(1,0)
(a,0)
(a+b,0)
u

(c) I On the coordinate axes, construct a line $\perp$ to this X-axis at (1,a). Call the line ℓ_1; mark off on ℓ_1 a segment of length a, with one end point at (1,0).

II Draw the line through the origin containing (1,a). Call it k. The slope of k is a and its equation is $y = ax$.

III Construct a $\perp$ to the Y-axis at (0,1); call the line ℓ_2.

IV Label the point of intersection of ℓ_2 and k Q.

V Drop a perpendicular from Q to the X-axis. Label the foot of this perpendicular on the X-axis P. $QP = 1$ and the length of the segment between (0,0) and P is b. Thus $a \cdot b = 1$. The property illustrated is: Every non-zero real number has a multiplicative inverse (or reciprocal).

9-5 Reading a Table

One of the most useful skills to be learned in Junior High School Mathematics is the ability to read a mathematical table. Among tables which may be introduced at this level are tables of trigonometric ratios, simple interest, and squares and square roots. No work in interpolation in tables is expected in this course. While no use is made of the cotangent ratio it is desirable to include this ratio to make the usual table a little more complete. It also gives the teacher an opportunity to explain the co-function relation with two examples rather than one.

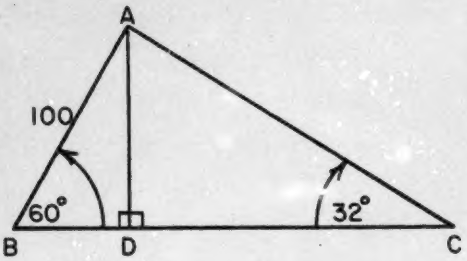
Since angles of measure 0 have not been defined in this book and since we are restricting our attention to acute angles contained in right triangles the usual lines in the table for 0° and for 90° are omitted. If the teacher wishes to discuss $\sin 0^\circ$ as a limit for $\sin x$ as x approaches 0, no harm will be done. It may be appropriate to observe that tan x, as x approaches 90°, increases without limit.

The approximate nature of the numbers, correct to ten-thousandths, to be found in the table should be emphasized. It might be pointed out that this table was not computed from measurements (although this would be theoretically possible) and that in advanced courses other ways of constructing such a table will be studied.

Answers to Exercises 9-5

1. (a) 0.1736
   (b) 0.1763
   (c) 0.6561
   (d) 0.8910
   (e) 0.9903
   (f) 0.8391
   (g) 1.1918
   (h) 1.7321
   (i) 2.7475
   (j) 0.9994
2. A. yes
   B. yes
   C. yes
   D. yes
   E. yes
3. The tangent of an angle is greater than zero but not always less than one. The tangent of an angle increases with the size of the angle between 1° and 89° . The differences between consecutive table readings varies throughout the table. The differences between the tangents of two consecutive angles are smaller for smaller consecutive angles than for larger consecutive angles.
4. (a) 52.99
   (b) 89.9586
   (c) 0.2582|01 or 0.2582
   (d) 0.2835|65 or 0.2836

*5.



In right triangle ABD

$\frac{AD}{AB} = \text{Sin } 60^\circ$

$AD = 100 \text{ Sin } 60^\circ$

$AD = 100(0.8660) = 86.6$

$= 87$

to the nearest integer.

In right triangle ADC

$\frac{AD}{DC} = \text{Tan } 32^\circ$

$\frac{DC}{AD} = \text{Cot } 32^\circ$

$DC = (86.6)(1.6003)$

DC = 138 to the nearest integer.

*6. (a) (1) $\cot \angle POQ = \tan \angle PQO = \frac{OP}{PQ}$.

(2) $\tan \angle POQ = \frac{PQ}{OP}$.

(3) $\frac{1}{\tan \angle POQ} = \frac{1}{\frac{PQ}{OP}} = 1 \div \frac{PQ}{OP} = 1 \cdot \frac{OP}{PQ} = \frac{OP}{PQ}$.

(4) By (1) and (3), $\cot \angle POQ = \frac{1}{\tan \angle POQ}$.

(b) No. $\sin \angle POQ = \frac{PQ}{OQ}$ and $\cos \angle POQ = \frac{OP}{OQ}$.

These are not reciprocals of each other.

9-6 Kinds of Variation

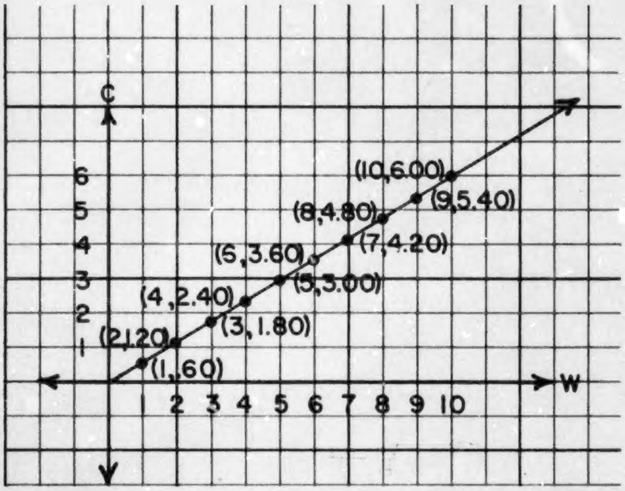
The concept of variation can be very appropriately introduced by use of tables of data and graphs. An attempt has been made in this section and throughout the chapter to use examples from science. It is hoped that one of the contributions of these two books for use in Grades 7 and 8 can be the encouragement of the practice of using applications of the mathematics to science as well as to the more traditional business and social topics.

Here several types of variation are used so that some differences among them can be observed. In the sections to follow specific attention will be given to direct and inverse variation and in an optional section to still other topics. Attention is called to the graph of $y = ax$ to show a relation of the second part of the chapter beginning with this section to the first part of the chapter on similar right triangles.

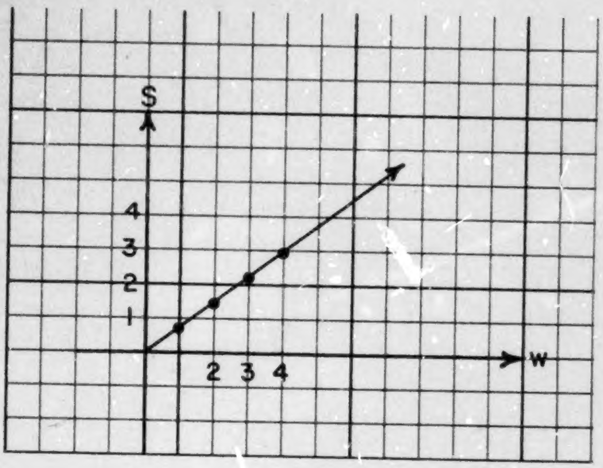
Answer to Exercises 9-6

1.

w	Amount in pounds	0	1	2	3	4	5	6	7	8	9	10
c	Cost in dollars	$ 0	$ .60	$ 1.20	$ 1.80	$ 2.40	$ 3.00	$ 3.60	$ 4.20	$ 4.80	$ 5.40	$ 6.00



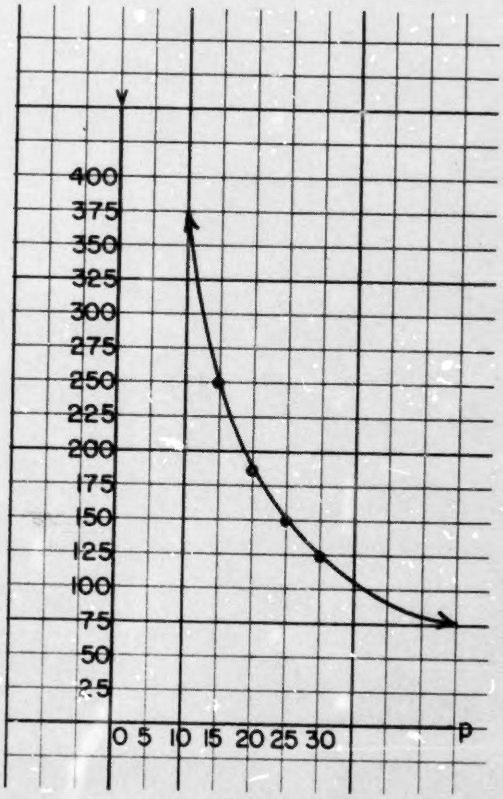
2. (a)



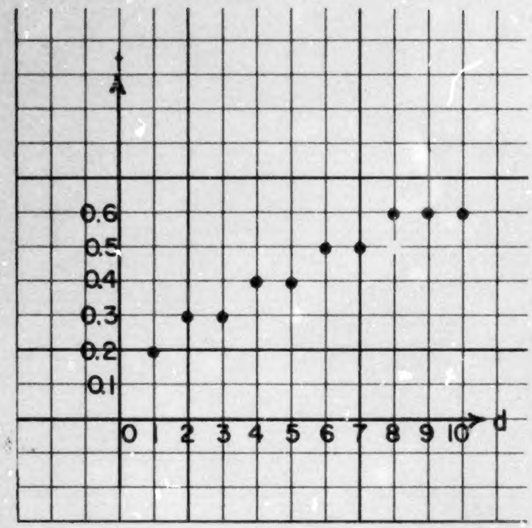
(b) $s = \frac{3}{4}w,\ s>0,\ w>0$

(c) $s = \frac{3}{4}(2\frac{1}{2}) = 1\frac{7}{8}$.

3. If $P = 40$, $v \approx 94$



4.

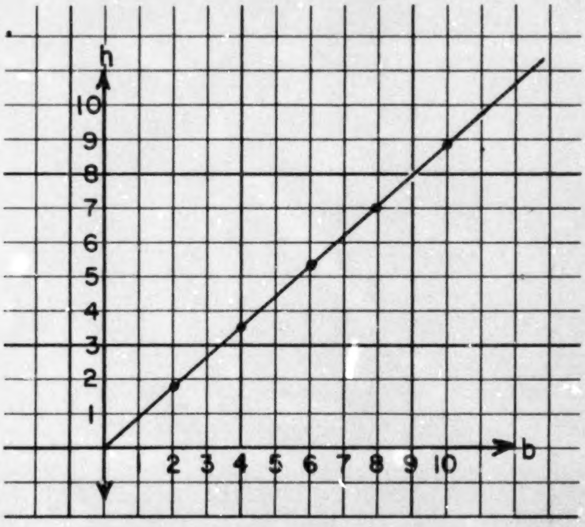


The prediction can safely be stated as more than 0.6 second.

5. Using the relation $h = \frac{b}{2}\sqrt{3}$ we have:

b	2	4	6	8	10
h	$\sqrt{3}$	$2\sqrt{3}$	$3\sqrt{3}$	$4\sqrt{3}$	$5\sqrt{3}$
	1.7	3.5	5.2	6.9	8.7

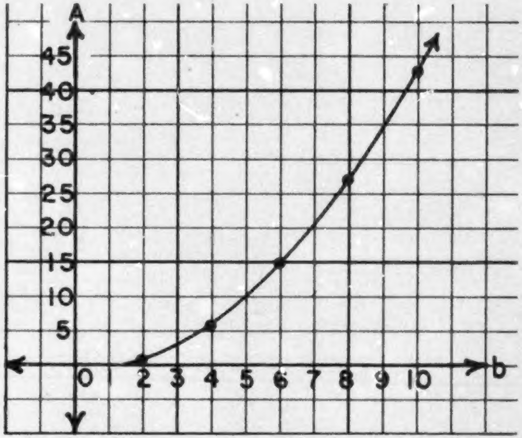
The graph is a half line.



6. $h = \frac{b}{2}\sqrt{3}, \; b>0$

7.

b	2	4	6	8	10
A	$\sqrt{3}$	$4\sqrt{3}$	$9\sqrt{3}$	$16\sqrt{3}$	$25\sqrt{3}$
	1.7	6.9	15.6	27.7	43.3



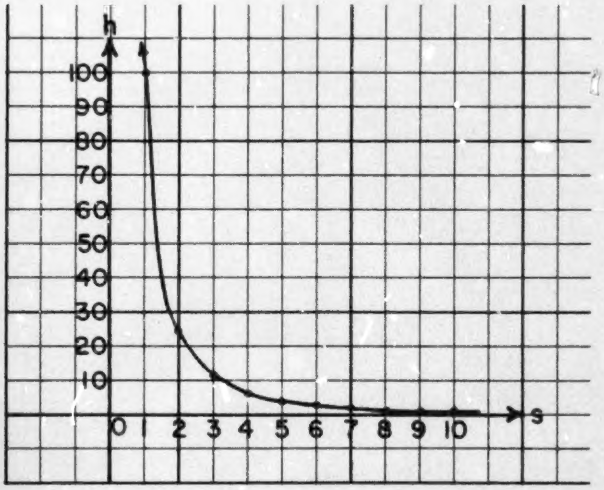
The graph is the right half of the parabola

$A = \frac{b^2\sqrt{3}}{4}, \quad b>0.$

8.

s	1	2	3	4	5	6	7	8	9	10
h	$\frac{100}{1}$	$\frac{100}{4}$	$\frac{100}{9}$	$\frac{100}{16}$	$\frac{100}{25}$	$\frac{100}{36}$	$\frac{100}{49}$	$\frac{100}{64}$	$\frac{100}{81}$	$\frac{100}{100}$
	100	25	11.1	6.2	4	2.8	2.0	1.6	1.2	1

$h = \frac{100}{s^2}, \quad s \neq 0$



9-7 Direct Variation

One of the ways to talk about direct variation is through reference to the graph of a straight line. As x increases, y increases. This is true when negative values are considered for x as well as positive values. The increase in y for an increase of 1 in x is constant. This constant can be called the constant of proportionality. The constant of proportionality is also the ratio of the change in y, between two points on the graph of a straight line or between two of the elements in a set of data, to the change in x. If a value of y is multiplied by k the value of the corresponding x of an ordered pair (x, y) is also multiplied by k. The relation of direct variation can be expressed by the number sentence $y = ax$ for $a \neq 0$. Except for the number pair (0,0) the ratio of number pairs, $\frac{y}{x}$, which satisfies the equation, is a. Since for number pairs in which the relation is direct variation, $\frac{y}{x} = a$, the set of numbers y and the set of numbers x are said to be proportional.

Properties like these enumerated in the above paragraph can be discussed carefully and in some detail for direct variation. Then in considering inverse and other kinds of variation it may be helpful to test these properties to see if they hold in the new situation.

Answers to Questions in 9-7

Multiply, the answer is $9.00.

$c = (0.60)(w)$.

Slope is 0.60. Now $(0.60)(10) = 6.00$ which is the increase in cost.

$\frac{c}{w}$ is the change in cost for each unit change in weight.

(50)(1) = 50: 50 miles in one hour.

(50)(2) = 100: 100 miles in two hours.

(50)(3.5) = 175: 175 miles in three and one half hours.

(50)(t) = 50t: 50t miles in t hours.

$3 = (k)(2)$; $k = \frac{3}{2}$ or $1\frac{1}{2}$.

$s = kw$.

$s = (\frac{3}{2})(5) = 3(2\frac{1}{2}) = 7\frac{1}{2}$. The stretch produced by a five pound weight is $7\frac{1}{2}$ inches.

Answers to Exercises 9-7

1. $t = 32n$
   $d = .32n$ or $\frac{32}{100}n$ or $\frac{8}{25}n$
2. $c = 33.9\ g$
3. $d = 2n$
   no.
4. $i = 12f$
   i increases.
5. $k_1 = 32$ if c is in terms of cents. $.32$ if c is in dollars.
   $k_2 = 33.9$
   $k_3 = 2$
   $k_4 = 12$
6. Yes, if $n \neq 0$
7. $y = kx$
   $6 = (k)(2)$
   $3 = k$

8. $y = kx$

   $^-3 = (k)(^-12)$

   $\frac{1}{4} = k$

9. Yes, when b increases, a increases, when b decreases, a decreases. $a = 10b$

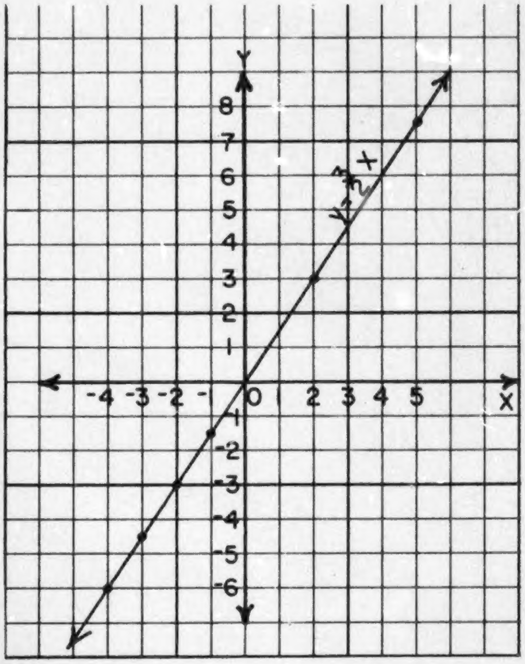
10. $d = kt$

    $240 = (k)(6)$

    $40 = k$

    $\therefore\ d = 40t$

11. $(^-4, ^-6)$, $(^-3, ^-4\frac{1}{2})$, $(^-2, ^-3)$, $(^-1, ^-1\frac{1}{2})$ $(0,0)$, $(2, 3)$, $(5, 7\frac{1}{2})$.



12. (a) Yes, y is doubled.
    (b) y is halved.
    (c) x is multipled by 10.
    (d) Yes.

13. x is halved.
    y is tripled.

14. Sample: $f = 5280\,m$, where m is the number of miles and f is the number of feet.
    $M = \left(\frac{1}{100}\right)(c)$, where c is the number of centimeters and M is the number of meters.

9-8 Inverse Variation

In the study of any concept in mathematics it is desirable to observe situations in which the properties associated with the concept do not hold as well as to study situations in which the properties hold. For this reason, if no other, it is highly desirable to introduce inverse as well as direct variation. In addition there are many important relations in science and in social situations which are illustrations of inverse variation. A number of these are included in the exercises.

It has been said that the arithmetic of Grades 1 through 4 is concerned with the relation $x + y = a$, while the arithmetic of Grades 5 through 8 is concerned with the relation $xy = a$. This statement is based on the important place in upper grade mathematics of such topics as multiplication and division, area, per cent problems, and rate problems. Interesting interpretations of inverse variation may be made in any of these situations.

This relation of inverse variation can be expressed by the number sentence $xy = a$, $a \neq 0$, or by $y = \frac{a}{x}$. In either case no value can be obtained for y if x is 0, or for x if y is 0. This introduces a new situation with graphs. The graph of $xy = a$ does not cross the Y-axis or the X-axis. Hence the graph is in two parts and is said to be discontinuous. It may be well to point out that y can be determined for x any positive number, no matter how small, even for $x = 10^{-1000}$. Similarly, a value can be determined for y if x is any negative number, no matter how close the number is to zero. In higher mathematics the students will learn that the lines $x = 0$ and $y = 0$ are asymptotes of the hyperbola $xy = a$ and that the hyperbola is said to approach the axes asymptotically.

Answers to Questions in Sec. 9-8

As the number of guests increases, the amount of punch per guest decreases; the amount is halved.

$$p = \frac{10}{n},\ n > 0.$$

the amount of punch per person is $(\frac{1}{3})(p)$ if n is tripled.

In Joel's experiment where $pv = k$, $k = 3750$. This does fit his observations.

Answer to Exercises 9-8

1. (a)

r(number of miles per hr.)	10	20	25	50	60	75	80	100
t(number of hours)	10	5	4	2	$1\frac{2}{3}$	$1\frac{1}{3}$	$1\frac{1}{4}$	1

(b) $(r)(t) = 100$

(c) The time is halved.

(d) r decreases, such that the product of r and t is still 100.

2. (a)

s	10	12	15	16	30	40
n	24	20	16	15	8	6

(b) $(n)(s) = 240$ where both n and s are counting numbers only.

3. $WD = 36$

(a)

W	2	9	18	36	6	4
D	18	4	2	1	6	12

(b) If D is doubled, W is halved.
As W increases, D decreases such that the product of W and D is still 36.

4. $(r)(p) = 200$
If the number p is doubled, the number r is halved.
If the number r is doubled, the number p is halved.

5. $k_1 = 100$

$k_2 = 240$

$k_3 = 36$

$k_4 = 200$

6. For direct variation the ratio of the variables is constant. For inverse variation, the product of the variables is a constant.

7. $xy = k$
$(2)(6) = 12$
$k = 12$

8. $xy = k$

$(\frac{1}{2})(10) = k$

$\therefore\ k = 5$

9. No. The product of a and b is not a constant. The ratio $\frac{a}{b}$ is a constant. The table suggests that a and b are related by the equation $a = \frac{1}{2}b$.

10. (a) No.

(b) No. They are related by $y = 2x^2$.

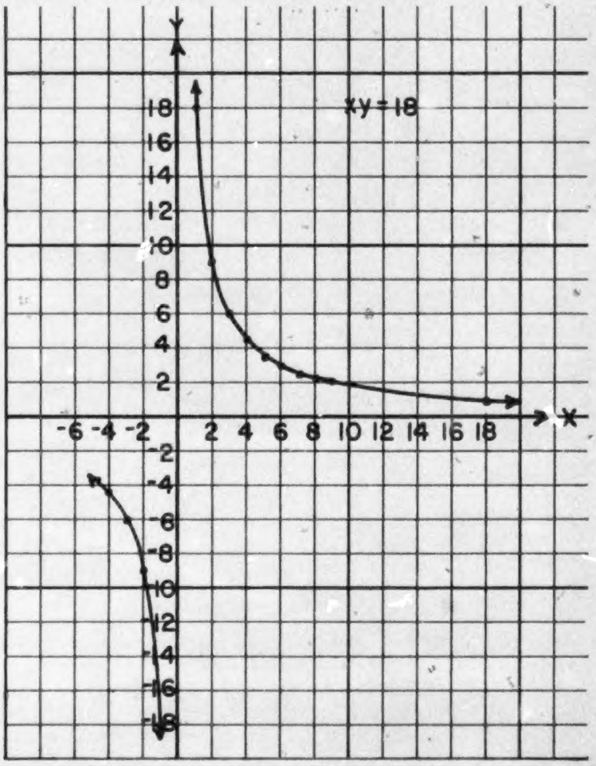
11. (a)

x	$^{-}4$	$^{-}3$	$^{-}2$	$^{-}1$	1	2	3	4	5	6	7	8	9	18
y	$^{-}4\frac{1}{2}$	$^{-}6$	$^{-}9$	$^{-}18$	18	9	6	$4\frac{1}{2}$	$3\frac{3}{5}$	3	$2\frac{4}{7}$	$2\frac{1}{4}$	2	1

(b) No. If either factor is zero, the product would be zero. Since the product cannot be zero, neither factor can be zero.

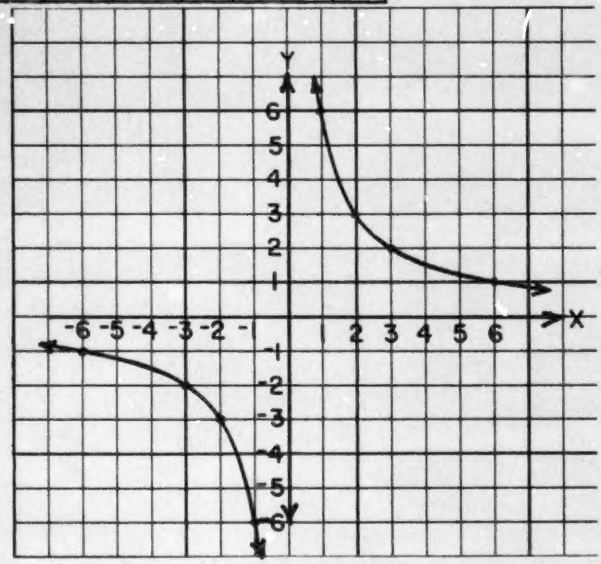
(c) Yes

(d) Inverse.



12. $xy = 6$

x	$^{-}6$	$^{-}3$	$^{-}2$	$^{-}1$	1	2	3	6
y	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}6$	6	3	2	1



9-9 (Optional) Other Types of Variation

In better classes particularly, the teachers may wish to give at least brief attention to other types of variation. For this reason variation of one number in an ordered pair as the square of the other number and, analogously, variation of one of a pair, "inversely as the square" of the other are introduced. Also reference is made to the graph of $y = \sin x$ for $x = 10, 20, \ldots, 80$. This last topic might be considered in order to refer again to some of the material earlier in the chapter. Unfortunately, students do not have at this time the necessary background for the study of the interesting and exceedingly important properties of the graph of $y = \sin x$.

This section might be assigned for self study by the students who have made the most rapid progress with the earlier parts of the chapter, or to those students who show the greatest interest in future scientific study.

Answers to Questions in 9-9 (Optional)

If the number t is doubled, the number d is multiplied by 4. If t is tripled, d is multiplied by 9.

$k = 16$

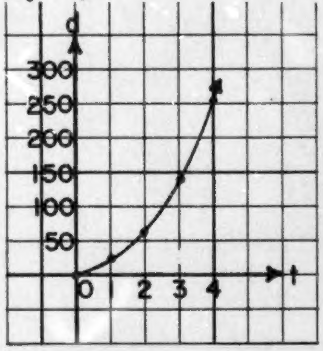
$d = 16t^2$

$d = 16(10)^2$

$d = 1600$

$\therefore$ the body falls 1600 ft. in 10 seconds.

Galileo: Born 1564 - Died 1642.



Answers to Exercises 9-9a

1. (a) 4 sq. in., 6 faces, 24 sq. in.

(b)

S	1	2	3	4	5	6
A	6	24	54	96	150	216

2. (a) $S = e^2$

(b) S varies as the square of e.

(c)

e	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225

(d) (1) If $e = 3$, $S = 9$

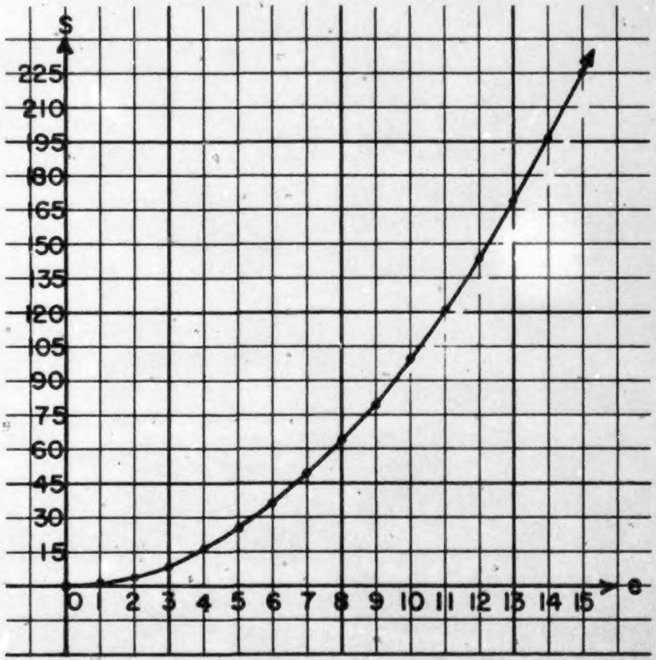
(2) If $S = 64$, $e = 8$

(3) If $e = 5.5$, $S \approx 30$

(4) If $S = 40$, $e \approx 6.4$

(e) (1) If $e = 3$, $S = 9$

(2) If $e = 5.5$, $S = 30.25$



3.

d	1	2	3	4	5	6	7	8	9	10
t	0.2	0.3	0.3	0.4	0.4	0.5	0.5	0.6	0.6	0.6
$\frac{d}{t^2}$	25	22.2	33.3	25	31.2	24	28	22.2	25	27.7

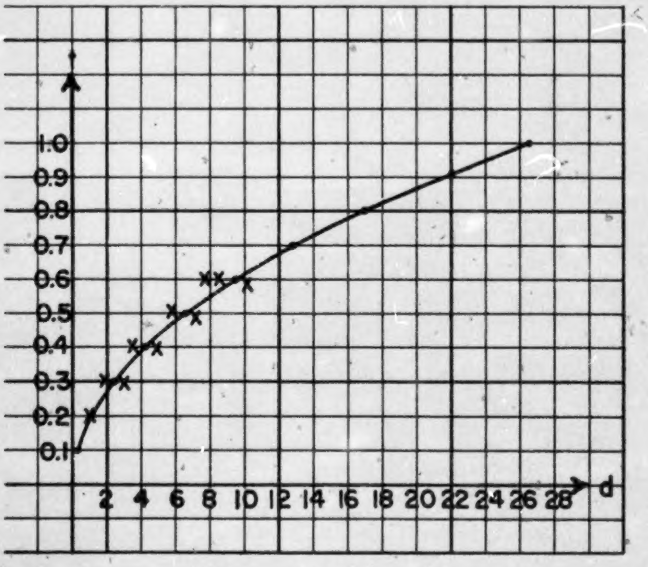
The mean of the ratios $\frac{d}{t^2}$ is 26.4

$d = 26.4t^2$

d	0.3	1.0	2.4	4.2	6.6	9.5	12.9	16.9	21.4	26.4
t	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0

Yes, the theoretical curve fits.

Yes $\frac{d}{t^2}$ seems to remain constant according to her data.



x marks Phyllis' observations
. mark points described by
$d = 26.4t^2$.

4. $E = kV^2$

$64 = k(4)^2$

$4 = k$

(a) $E = 4V^2$

(b) $E = 4(6)^2 = 144$

(c) $16 = 4V^2$

$4 = V^2$

$V = 2$

5. (a) $\frac{100}{280} = \frac{x}{1}$, $x = \frac{5}{14}$ $\therefore \frac{5}{14}$ lbs. is required.

(b) $\frac{5}{14} \times (70) = 25$, $\therefore$ 25 ¢ is required.

(c) $C = KS^2$ $K = \frac{70}{280} = \frac{1}{4}$; $C = \frac{1}{4}S^2$

(d) $C = \frac{1}{4}(65)^2$

$C = 1056\frac{1}{4}$ ¢ or cost = \$10.57.

(e) $C = \frac{1}{4}\,S^2$

$C = \frac{1}{4}(75)^2 = 1406\frac{1}{4}$, Cost = \$14.07. Enough can be bought to sow the plot.

6. (a) $d = kt^2$

(b) $d = 9k$

(c) $144 = 9k$

$16 = k$

$d = 16t^2$

(d) $d = 16(5)^2$

$d = 400$ $\therefore$ the distance is 400 ft.

Answers to Exercises 9-9b (Optional)

1. $P = 4e$

   $S = 6e^2$

   $V = e^3$

2. (1) P varies directly as e.

   (2) S varies as the square of e.

   (3) V varies as the cube of e.

3.

e	0	1	2	3	4	
P	0	4	8	12	16	for $P = 4e$
S	0	6	24	54	96	for $S = 6e^2$
V	0	1	8	27	64	for $V = e^3$

4. If $(e) = 2\frac{1}{2}$, $P = 10$

   $S = 37\frac{1}{2} \approx 38$

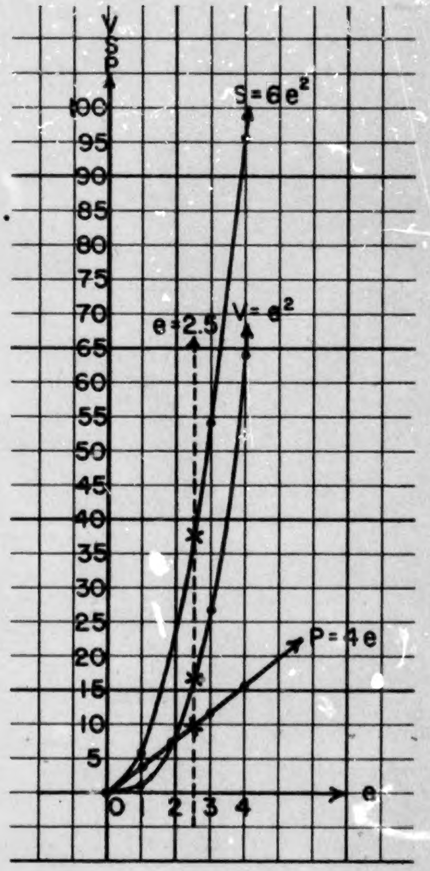
   $V = 15.625 \approx 16$

5. If $e = 10$, $P = 40$

   $S = 6(10)^2 = 600$

   $V = (10)^3 = 1000$

   $\therefore$ if $e = 10$, $P < S < V$



9-10 Summary

The committee which prepared this chapter has no new recommendation on the use of the summary. It is included for the convenience of teacher and students. The review exercises could be used as test questions, additional assignments, or general class summary discussion.

Answers to Exercises 9-10

1. $y = kx$

$16 = k(2)$

$\therefore k = 8$ and $y = 2x$

$y = 2x$

$y = 2(5) = 10.$

2. $xy = k$

$(2)(16) = k$

$k = 32$

$\therefore xy = 32$

$(5)(y) = 32$

$y = 6\frac{2}{5}$

3. $y = kx^2$

$16 = k(2)^2$

$k = 4$ and $y = 4x^2$

$y = 4(5)^2 = 100$

4. $\frac{A_1}{A_2} = \frac{{d_1}^2}{{d_2}^2} = \frac{(4)^2}{(6)^2} = \frac{16}{36} = \frac{4}{9}$

5. (a) $d = kp$

$5 = k(10)$

$k = \frac{1}{2}$ and $d = \frac{1}{2}p$

(a) $14 = \frac{1}{2}p$

$\therefore p = 28$, pull must be 28 [illegible].

(b) $d = \frac{1}{2}(14) = 7$, distance will be 7 in.

6. $\frac{1}{2}$ pint = $\frac{1}{16}$ gal.

$\therefore$ 7 lbs. per sq. in. for 1 gal. (7)(16) lbs. per sq. in for $\frac{1}{2}$ pint.

or

$PV = K$

$(7)(1) = K = 7$. If $V = \frac{1}{16}$, then $P(\frac{1}{16}) = 7$

$P = (16)(7) = 112$, hence, 112 lbs. per sq. in. is the pressure exerted by a certain amount of hydrogen in a 1 gal. jug if it is later enclosed in a half-pint jar.

7. In $x + y = k$, y does not vary inversely as x because the product of x and y is not constant.

8. Inverse variation, $xy = k$

9. Direct variation $y = kx$
π is the constant of proportionality.

10. $V = \pi r^2 h$

If r is multiplied by 5 and h is unchanged

$V = \pi(5r)^2 h$

$\therefore$ $V = 25(\pi r^2 h)$

hence the volume is multiplied by 5.

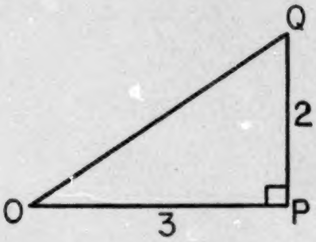
Sample Test Questions

The following questions are not intended to provide a single test for Chapter 9. They are intended only to help the teacher in preparing his own tests.

Part I. True - False

(T) 1. Triangle ABC is similar to triangle A'B'C' if $\frac{AB}{A'B'} = \frac{BC}{B'C'}$ and $m(\angle ABC) = m(\angle A'B'C')$

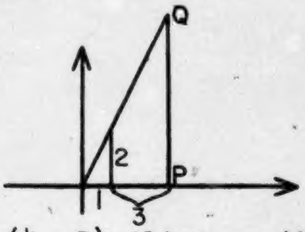
(F) 2. In the figure



$\sin \angle POQ = \frac{2}{3}$

(F) 3. $\text{Sin } 40^\circ = \frac{1}{\cos 40^\circ}$

(F) 4. In the figure below PQ = 6.



(F) 5. The point (4, 5) lies on the line whose equation is $y = \frac{4}{3}x$.

(T) 6. The point (4, 5) lies on the line joining (0, 0) and (12, 15).

(F) 7. The tangent of an angle is always a number between 0 and 1.

(T) 8. The height of a tree may be calculated if the length of its shadow and the angle of elevation of the sun are known.

(F) 9. Any two right triangles are similar.

(T) 10. If $F = 14a$, F varies directly as a.

(F) 11. If $F = \frac{k}{6}$, F varies inversely as 6.

(T) 12. If x varies inversely with y and y is 4 when x is 9, then $y = 2$ when $x = 18$.

(F) 13. If y varies directly as x .then x varies inversely with y.

(T) 14. If the measures of two angles of one triangle are equal, respectively, to the measures of two angles of another triangle, then the third angles also have equal measure.

(F) 15. If two sides of one triangle have lengths equal, respectively, to the lengths of two sides of another triangle, then the third sides must also have equal lengths.

(F) 16. The sine of an angle varies directly with the measure of the angle.

Part II. Multiple-Choice

Write the letter of the best answer.

(c) 1. If one acute angle of a right triangle has the same measure as an acute angle of another right triangle, then the triangles are

(a) congruent
(b) isosceles
(c) similar
(d) all of the above
(e) cannot tell from this information

(d) 2. The lengths of the sides of a triangle are 6, 8, and 9 inches. Which of the following are the lengths of the sides of a triangle similar to it?

(a) 3, 4, 6 inches
(b) 2, 3, 4 inches
(c) 9, 12, 16 inches
(d) 18, 24, 27 inches
(e) none of the above

(c) 3. The equation of the line through (0, 0) and (4, 5) is

(a) $x = \frac{5}{4}y$
(b) $y = \frac{4}{5}x$
(c) $y = \frac{5}{4}x$
(d) $x = \frac{4}{5}y$
(e) none of the above

(c) 4. The slope of the line whose equation is $y = x$ is

(a) 0
(b) x
(c) 1
(d) it does not have any
(e) none of the above

(a) 5. The slope of the X-axis is

(a) 0
(b) 1
(c) $y = 0$
(d) it does not have any
(e) none of the above

(d) 6. The slope of the Y-axis is

(a) 0

(b) 1

(c) $x = 0$

(d) it does not have any

(e) none of the above

(c) 7. The sine of an angle of a right triangle

(a) depends on the length of one side only

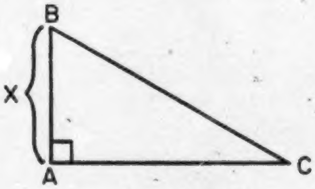
(b) depends on the length of the hypotenuse only

(c) depends on the measure of the angle only

(d) all of the above

(e) none of the above

(d) 8. In the drawing we can find the number x if we know



(a) AC and BC

(b) AC and m(∠ ACB)

(c) AC and m(∠ ABC)

(d) all of the above are correct

(e) none of the above is correct

(b) 9. If you spend two dollars buying beans, the cost per pound of the beans and the number of pounds purchased provide an example of

(a) direct variation
(b) inverse variation
(c) variation as the square
(d) inverse square variation
(e) none of the above

(b) 10. If p varies inversely with t and $p = 6$ when $t = 2$ then what is p when $t = 10$?

(a) 30

(b) $\frac{6}{5}$

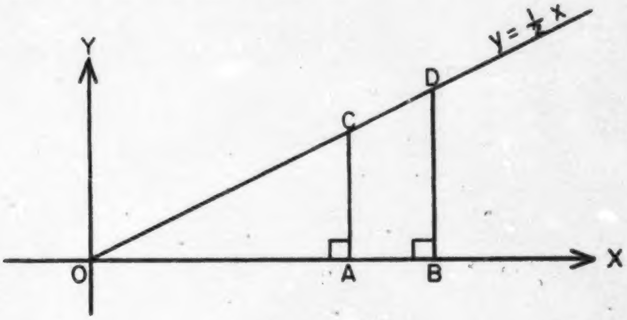
(c) $\frac{6}{10}$

(d) $\frac{5}{6}$

(e) not enough information

Part III. Completion

For questions 1-6 use this figure:



$\frac{5}{2}$ 1. The coordinates of point C are (5, _____).

8 2. The coordinates of point D are (____, 4)

$\angle$ ODB 3. m($\angle$ OCA) = m($\angle$____)

$\frac{1}{2}$ 4. tan $\angle$ AOC = __________.

$\frac{1}{\sqrt{5}}$ 5. sin $\angle$ AOC = __________.

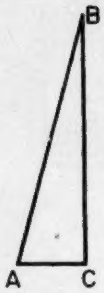
$\frac{1}{\sqrt{5}}$ 6. cos $\angle$ ACO = __________.

90 45 45 7. The measure of each of the three angles of an isoceles right triangle are _____, _____and______.

75 8. A right triangle has a 15° angle. The measure of the other acute angle is__________.

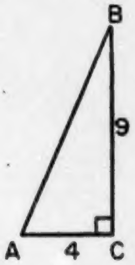
2 9. The slope of the line joining the points (2, 1) and (4, 5) is_________.

152 10. In the figure



tan $\angle A$ = 3.8 and $\overline{AC}$ has length 40 feet.
BC = ___________.

For questions 11 - 15 use this figure:



$\frac{9}{4}$ 11. tan $\angle A$ = __________.

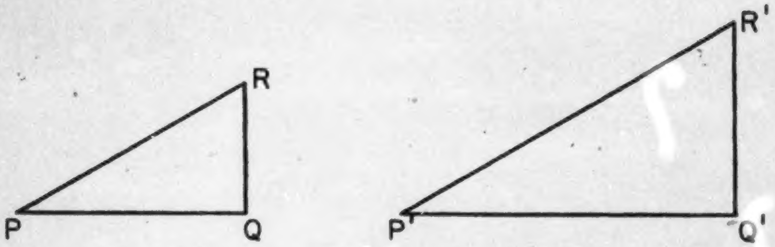
$\frac{9}{\sqrt{97}}$ 12. sine $\angle A$ = __________.

$\frac{4}{9}$ 13. tan $\angle B$ = __________.

$\frac{9}{\sqrt{97}}$ 14. cos $\angle B$ = __________.

$\frac{4}{9}$ 15. cot ∠ A = ________.

For questions 16 and 17 use this figure:



m(∠ Q) = m(∠ Q') = 90 and m(∠ A) = m(∠ A').

16 16. If QR = 12, Q'R' = 15, and P'Q' = 20 then PQ = __________.

30 17. If QR = 18, Q'R' = 27, and PR = 20 then P'R' = ________.

57.73 18. In the drawing

Given:

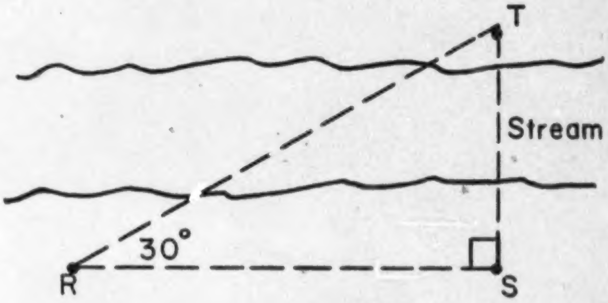
sin 30° = .5000

$\tan 30^\circ \approx \frac{1.7321}{3}$

= .5773.

$\cos 30^\circ \approx \frac{1.7321}{2}$

= .8660



S and T represent trees on opposite sides of the stream. The distance between R and S is 100 feet. The distance between S and T is_________feet.

115.46 19. In Problem 18, RT = ________.

inversely 20. If $y = \frac{\pi}{x}$ then y varies _________with x.